NON-CANONICAL EXTENSION OF θ-FUNCTIONS AND MODULAR INTEGRABILITY OF ϑ-CONSTANTS

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Abstract. We present new results in the theory of the classical θ-functions of Jacobi: series expansions and defining ordinary differential equations (ODEs). Proposed (Hamiltonian) dynamical systems define fundamental differential properties of theta-functions and yield an exponential quadratic extension of the canonical θ-series. An integrability condition of these ODEs explains appearance of the modular ϑ-constants and differential properties thereof. General solutions to all the ODEs are given. For completeness, we also solve the Weierstrassian elliptic modular inversion problem and consider its consequences. As a nontrivial application, we apply proposed technique to the Hitchin case of the sixth Painlevé equation.

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Contents

1. Introduction

The theta-functions of Jacobi and Weierstrassian basis of functions $\{\sigma, \zeta, \wp, \wp'\}$ arise in numerous theories and applications. Since their discovery in the late 1820's, this field became the subject of intensive study. The majority of results and current form of the theory were obtained in the very works of Jacobi and Weierstrass, and their contemporaries: Hermite [44], Enneper [30], Kiepert, Neumann [58], Halphen [40], Hurwitz [46], Frobenius [37], Fricke [36, 49] et al. Presently, only monographic literature on this topic may run into tens of items. Thorough treatises had appeared as far back as Jacobi's life [39] (not only in German [67]) and, continuing the list further, we should mention books [72, 18, 27, 34, 41, 50, 52, 65, 70, 75], references in these works, and especially encyclopedic paper by Fricke [35]. See also the excellent history survey by Koenigsberger [51]. Literature given in the reference list does not claim on completeness, but comprehensively contains properties of

elliptic, modular, and θ-functions known nowadays. In addition to monographs listed above it should be mentioned later presentation of the theory [64, 5, 54, 33, 63, 47, 57, 6, 16] and handbook literature as well [76, 4, 32]. A great number of specific examples can be found in the classical *'A course of modern analysis'* by Whittaker & Watson [79] and very detailed exposition of the theory is in Weber's *'Lehrbuch der Algebra'* [75] and in the two-volume Halphen treatise with the posthumous issue of the 3rd volume [40]. As formulae source, in most of cases Schwarz's collection of Weierstrass' and Jacobi's classical results [76] is by no means lacking and the four volume set by Tannery & Molk [69] hitherto contains most exhaustive information along these lines. The very Weierstrass' [77] and Jacobi's [48] works, being very detailed in presentation, should be referred hereinto since they have still remained a source of important observations.

1.1. **The paper content and comments to results.** In the present work we shall describe some new properties of Jacobi's θ-series, that, to the best of our knowledge, have not appeared in the numerous literature on this topic. Among these are series expansions, differential equations, and their consequences. A characteristic property of the Jacobi–Weierstrass theory is explicit analytic $\theta, \wp$-formulae for solutions of applied problems. In this connection we shall exhibit some examples: modular inversion problem, differential computations of Weierstrassian functions, and applications to the famous sixth Painlevé transcendent (Sect. 10). Other applications recently announced in [12, 13, 14].

1.1.1. *The power series.* The series expansions of elliptic, modular, and θ-functions are an ever-operating instrument in many problems up to now. This is because coefficients of the series have nice analytic and combinatoric properties. Suffice it to mention applications of the function series for various θ-quotients [75, 69], number-theoretic q-series, series of Lambert [6], the famous McKay–Thompson series and their corollaries like 'Moonshine Conjecture' and its modern extensions. Trigonometric series for Jacobi's θ-functions are put for their definition (see Sect. 2.1) and power series for Weierstrass' σ-function is very well known [76, 4]. It is frequently reproduced in the literature and has multidimensional generalizations [28]. In this connection it is somewhat surprise fact that the power series expansions for θ-functions, i. e., θ-analogs of Weierstrassian σ-series, are absent heretofore. It is of interest to remark that even Jacobi attempted[1] to obtain that series and observed that their coefficients resulted in interesting dynamical systems. In Sect. 3 we construct the canonical power expansions for θ-functions in a (fast computable) form that has the simplest structure and is maximally effective from the analytic point of view.

1.1.2. *Differential equations.* In sects. 4–7 and 9 we expound the main material of the work. Namely, dynamical systems satisfied by θ-series, their integrability condition, non-canonical extension of the canonical θ-series, and modular integrability of ϑ-constants. We shall see that not only do elliptic functions satisfy ordinary differential equations (ODEs)[2], but θ-functions themselves also satisfy certain ODEs. These ODEs are of interest in their own rights if only because, it is more logical to consider *ODEs proper for the θ-functions as basic equations* (Sect. 4) rather than ODEs for elliptic functions. Moreover, such a viewpoint is natural in a more general pattern since elliptic functions are the subclass of Abelian elliptic integrals and the latter are expressible in terms of theta-functions. In Sect. 9 we

[1] In a 1828 letter to Crelle [48, **I**: p. 259–260], i. e., before appearance of the famous *Fundamenta Nova*.

[2] The $\wp$-equation of Weierstrass or Euler's equations for Jacobi's functions sn, cn, and dn.

shall also see that solutions of introduced ODEs have a remarkable consequence, namely, the exponential quadratic extension with additional parameters. Under certain parameters this non-canonical extension coincides with the case of canonical θ-series.

There is yet another point worth noting. Differential relations between θ-functions are sometimes present in old literature [8, 75, 69] but they are regarded there just as differential identities. However, the principal point is a *differential closedness of the finitely many θ-objects* and, together with θ-functions, the theta-derivative θ_1' should play independent part in the theory. Upon introducing this object, analytic and differential manipulations by theta-functions are not in essence distinct from that by elliptic ones or even elementary functions like sine-, cosine-functions.

1.1.3. *Algebraic integrability and Hamiltonicity.* One further remarkable property of the systems mentioned above is the fact that the known basic polynomial theta-identities between canonical θ-series are nothing but the specific values of algebraic integrals of the systems (Sect. 7.1). Clearly, we may take these integrals as Hamiltonians for these ODEs being regarded as dynamical systems. Such treatments have a large number of applications and intimate connection with the theory of integrable systems [9]. Complete Hamiltonian description will be, however, the subject matter of a separate work but one particular case is discussed at greater length in [13]. In this work we underlined (with some counterexamples) the need to distinguish differential identities and defining ODEs. Algebraic integrability of the 'θ-ODEs' is also used in [14] for a new treatment of the finite-gap spectral problems.

1.1.4. *Modularity and integrability conditions.* Yet another consequence of the 'differential viewpoint' is an automatic appearance of the 'modular objects' (Sect. 6). Differential properties of the 'modular part' of Jacobi's functions are more transcendental and closely related to the classical theory of linear ODEs with infinite groups of Fuchsian monodromies. Moreover, in the early 1990's M. Ablowitz & S. Chakravarty with coauthors [20, 3, 21, 68] observed the deep linkage between this theory and complete integrable equations. In the last two decades, in works of Harnad [42], McKay [43], Ablowitz et all [25, p. 573–589], [2], Ohyama [59], Hitchin [45], this field was substantially advanced and found nice applications known as monopoles [7], Chazy–Picard–Fuch's equations [68], cosmological metrics of Tod [73], Hitchin [45] etc. In Sect. 7.1 we shall give further explanations and see that 'modular integrability' of the classical ϑ-constants has the following characterization. It constitutes a compatibility condition of the linear heat equation $4\pi \mathrm{i}\theta_\tau = \theta_{zz}$ and, on the other hand, the quadrature integrable nonlinear 'z-equations' for the functions $\theta(z|\tau)$ and $\theta'(z|\tau)$.

It may be also mentioned here that proposed 'differential technique' may be applied equally well to the cases of multidimensional Θ-functions when the latter split into a decomposition of the 1-dimensional, i. e., Jacobi's θ-functions. It is known that such cases correspond to jacobians of algebraic curves covering the elliptic ones [9]. The problems, wherein that curves arise, are very nontrivial on the one part, and are completely at hand as the pure elliptic case on the other part.

1.1.5. *Modular inversion problem.* Complete form of the 'differential theory' requires a closed form solution to the Weierstrassian elliptic modular inversion problem. Strange though it may seem, but its formula realization does not appear in the literature and, in Sect. 8, we shall give that solution and exhibit some consequences and applications thereof.

Content of the next section is a matter of common knowledge and we present it here to fix notation and terminology.

2. Definitions and notation

2.1. **The Jacobi functions.** The four theta-functions $\theta_{1,2,3,4}$ and their $\theta_{\alpha\beta}$-equivalents are defined by the following canonical series:

$$\begin{aligned}
-\theta_{11}(z|\tau) \equiv \theta_1(z|\tau) &= -\mathrm{i}\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}\tau}\sum_{-\infty}^{+\infty}{}_{k}(-1)^k\,\mathrm{e}^{(k^2+k)\pi\mathrm{i}\tau}\,\mathrm{e}^{(2k+1)\pi\mathrm{i}z}\\
&= 2\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}\tau}\sum_{0}^{\infty}{}_{k}(-1)^k\,\mathrm{e}^{(k^2+k)\pi\mathrm{i}\tau}\sin(2k+1)\pi\,z,\\
\theta_{10}(z|\tau) \equiv \theta_2(z|\tau) &= \mathrm{e}^{\frac{1}{4}\pi\mathrm{i}\tau}\sum_{-\infty}^{+\infty}{}_{k}\,\mathrm{e}^{(k^2+k)\pi\mathrm{i}\tau}\,\mathrm{e}^{(2k+1)\pi\mathrm{i}z}\\
&= 2\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}\tau}\sum_{0}^{\infty}{}_{k}\,\mathrm{e}^{(k^2+k)\pi\mathrm{i}\tau}\cos(2k+1)\pi\,z,\\
\theta_{00}(z|\tau) \equiv \theta_3(z|\tau) &= \sum_{-\infty}^{+\infty}{}_{k}\,\mathrm{e}^{k^2\pi\mathrm{i}\tau}\,\mathrm{e}^{2k\pi\mathrm{i}z}\\
&= 1+2\sum_{1}^{\infty}{}_{k}\,\mathrm{e}^{k^2\pi\mathrm{i}\tau}\cos 2k\pi\,z,\\
\theta_{01}(z|\tau) \equiv \theta_4(z|\tau) &= \sum_{-\infty}^{+\infty}{}_{k}(-1)^k\,\mathrm{e}^{k^2\pi\mathrm{i}\tau}\,\mathrm{e}^{2k\pi\mathrm{i}z}\\
&= 1+2\sum_{1}^{\infty}{}_{k}(-1)^k\,\mathrm{e}^{k^2\pi\mathrm{i}\tau}\cos 2k\pi\,z,
\end{aligned}$$

where, to avoid further confusion with Weierstrass' branch points e's, we use notation e^z for exponent of z. We shall use also the shorthand form $\theta_k := \theta_k(z|\tau)$. Values of θ-functions under $z=0$ (Thetanullwerthe [48]) are called the theta-constants. Because each of the variables z and τ has an 'independent theory' we introduce a separate notation for the nullwerthe: $\vartheta_k := \vartheta_k(\tau) = \theta_k(0|\tau)$. Their series representations follow from the θ-series above:

$$\vartheta_2(\tau) := \mathrm{e}^{\frac{1}{4}\pi\mathrm{i}\tau}\sum_{-\infty}^{\infty}{}_{k}\,\mathrm{e}^{(k^2+k)\pi\mathrm{i}\tau}, \quad \vartheta_3(\tau) := \sum_{-\infty}^{\infty}{}_{k}\,\mathrm{e}^{k^2\pi\mathrm{i}\tau}, \quad \vartheta_4(\tau) := \sum_{-\infty}^{\infty}{}_{k}(-1)^k\,\mathrm{e}^{k^2\pi\mathrm{i}\tau}. \tag{1}$$

For typographical convenience the double notation for θ-functions with characteristics $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ (Hermite (1858)) will be used:

$$\theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](z|\tau) \equiv \theta_{\alpha\beta}(z|\tau) = \sum_{-\infty}^{+\infty}{}_{k}\,\mathrm{e}^{\pi\mathrm{i}\left(k+\frac{\alpha}{2}\right)^2\tau+2\pi\mathrm{i}\left(k+\frac{\alpha}{2}\right)\left(z+\frac{\beta}{2}\right)}. \tag{2}$$

Let n, m be arbitrary integers: $n, m = 0, \pm1, \pm2, \ldots$. We consider only integral characteristics and hence, by virtue of formula

$$\theta\left[\begin{smallmatrix}\alpha+2m\\ \beta+2n\end{smallmatrix}\right] = (-1)^{\alpha n}\cdot\theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right], \tag{3}$$

functions $\theta_{\alpha\beta}$ always reduce to $\pm\theta_{1,2,3,4}$. When adding a half-period, θ-characteristics undergo a shift:

$$\theta[{\alpha \atop \beta}]\Big(z+\frac{n}{2}+\frac{m}{2}\tau\Big|\tau\Big)=(-\mathrm{i})^{(\beta+n)m}\,\theta[{\alpha+m \atop \beta+n}](z|\tau)\cdot\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}m(4z+m\tau)}. \tag{4}$$

Two-fold shifts by half-periods yield the law of transformation of θ-function into itself:

$$\theta_{\alpha\beta}(z+n+m\tau|\tau)=(-1)^{n\alpha-m\beta}\,\theta_{\alpha\beta}(z|\tau)\cdot\mathrm{e}^{-\pi\mathrm{i}m(2z+m\tau)}.$$

Value of any θ-function at any half-period is a certain ϑ-constant multiplied by the exponential factor:

$$\theta[{\alpha \atop \beta}]\Big(\frac{n}{2}+\frac{m}{2}\tau\Big|\tau\Big)=(-\mathrm{i})^{(\beta+n)m}\,\vartheta[{\alpha+m \atop \beta+n}](\tau)\cdot\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}m^2\tau}.$$

In the present work we use the 'τ-representation' for ϑ,θ-functions. Transition to frequently used 'q-representation' $\big(q=\mathrm{e}^{\pi\mathrm{i}\tau}\big)$ is performed by the formula $\partial_q=\pi\mathrm{i}q\partial_\tau$.

We supplement the set of functions $\{\theta_k\}$ with the derivative $\partial_z\theta_1(z|\tau)$ and consider it as a fifth independent object:

$$\begin{aligned}\partial_z\theta_1(z|\tau)=:\theta_1'(z|\tau)&=\pi\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}\tau}\sum_{k=-\infty}^{+\infty}(-1)^k\,(2k+1)\,\mathrm{e}^{(k^2+k)\pi\mathrm{i}\tau}\,\mathrm{e}^{(2k+1)\pi\mathrm{i}z}\\&=2\pi\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}\tau}\sum_{k=0}^{\infty}(-1)^k(2k+1)\,\mathrm{e}^{(k^2+k)\pi\mathrm{i}\tau}\cos(2k+1)\pi z.\end{aligned} \tag{5}$$

2.2. **The Weierstrass functions.** We use the conventional Weierstrassian notation [76]

$$\wp(z|\omega,\omega')=\wp(z;g_2,g_3),\qquad \wp'(z|\omega,\omega')=\wp'(z;g_2,g_3)$$

and the same notation of arguments for σ and ζ. Invariants (g_2,g_3) are functions of periods $(2\omega,2\omega')$ (and vice versa) and modulus $\tau=\omega'/\omega$. They are defined by the well-known Weierstrass–Eisenstein series [77, 29, 78] which are, however, entirely unsuited for numeric computations. Hurwitz, in his dissertation [46, p. 547], found a nice transition to the function Lambert series [6] which are used in theories, have applications, and are most effective in computations:

$$g_2(\tau)=20\pi^4\left\{\frac{1}{240}+\sum_{k=1}^{\infty}\frac{k^3\,\mathrm{e}^{2k\pi\mathrm{i}\tau}}{1-\mathrm{e}^{2k\pi\mathrm{i}\tau}}\right\},\qquad g_3(\tau)=\frac{7}{3}\pi^6\left\{\frac{1}{504}-\sum_{k=1}^{\infty}\frac{k^5\,\mathrm{e}^{2k\pi\mathrm{i}\tau}}{1-\mathrm{e}^{2k\pi\mathrm{i}\tau}}\right\}. \tag{6}$$

Determination of periods $(2\omega,2\omega')$ by coefficients (a,b) of elliptic curve in Weierstrassian form $w^2=4z^3-az-b$ is know as the elliptic modular inversion problem. Its solution involves the transcendental equation $J(\tau)=A$, where $J(\tau)$ is the classical modular function of Klein [49, 47, 5]. Modular inversion is then realized by the scheme

$$(a,b)\dashrightarrow J(\tau)=\frac{a^3}{a^3-27b^2}\dashrightarrow\omega=\pm\sqrt{\frac{a}{b}\frac{g_3(\tau)}{g_2(\tau)}}\dashrightarrow\omega'=\tau\omega\dashrightarrow(\omega,\omega'). \tag{7}$$

The degenerated cases—lemniscatic ($b=0$) and equi-anharmonic ($a=0$) ones—require separate formulae. In both of these cases there exist exact solutions. The lemniscatic solution ω_{L} was found by Gauss. In our notation it is as follows

$$\omega_{\mathrm{L}}=\sqrt[-4]{8a}\,\pi\cdot\vartheta_4^2(2\mathrm{i}),\qquad \omega'=\mathrm{i}\omega_{\mathrm{L}}.$$

See works by Todd [74] and Levin [55] for exhaustive information and voluminous bibliography on the lemniscate. Exact solution ω_{E} to the equi-anharmonic case we display here seems to be new:

$$\omega_{\mathrm{E}} = \sqrt[-12]{-27b^2}\,\pi\cdot\boldsymbol{\eta}^2(\varrho), \qquad \omega' = \varrho\omega_{\mathrm{E}}, \quad \varrho := -\frac{1}{2}\big(1-\mathrm{i}\sqrt{3}\big),$$

where $\boldsymbol{\eta}$ is the Dedekind function (see Sect. 2.3 for definition). The arbitrary branches of $\sqrt{\ }$-root are allowed in the previous formulae.

By virtue of homogeneity relations, say $\alpha^2\wp(\alpha z|\alpha\omega,\alpha\omega') = \wp(z|\omega,\omega')$, the couple of half-periods (ω,ω') or invariants (g_2,g_3) can be replaced by one quantity, i. e., modulus $\tau = \omega'/\omega$. We denote corresponding functions as follows:

$$\sigma(z|\tau) := \sigma(z|1,\tau), \quad \zeta(z|\tau) := \zeta(z|1,\tau), \quad \wp(z|\tau) := \wp(z|1,\tau), \quad \wp'(z|\tau) := \wp'(z|1,\tau).$$

Weierstrassian η-function is defined by the formula $\eta(\tau) = \zeta(1|1,\tau)$ and its series representation reads

$$\eta(\tau) = 2\pi^2\left\{\frac{1}{24} - \sum_{1}^{\infty}{}_{k}\, \frac{\mathrm{e}^{2k\pi\mathrm{i}\tau}}{(1-\mathrm{e}^{2k\pi\mathrm{i}\tau})^2}\right\}. \tag{8}$$

Modular transformations in the elliptic/modular theory are of not only theoretical interest since value of the modulus τ strongly affects convergence of the series. Moving τ into fundamental domain of the modular group

$$\tau\mapsto\frac{a\tau+b}{c\tau+d}, \qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) =: \boldsymbol{\Gamma}(1)$$

(this process is easily algorithmizable), one obtains values of τ having the minimal imaginary part $\Im(\tau) = \frac{1}{2}\sqrt{3}$. In such 'the most worst' point all the series converge very fast. For example modular property of the η-series is as follows

$$\eta\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^2\eta(\tau) - \frac{\pi}{2}\mathrm{i}c(c\tau+d),$$

where (a,b,c,d) are integers and, as usual, $ad-bc=1$.

The three Weierstrassian σ-functions are defined through Jacobian functions by the following expressions [69, 76]:

$$\sigma_\lambda(z|\omega,\omega') = \frac{\theta_{\lambda+1}\big(\frac{z}{2\omega}\big|\frac{\omega'}{\omega}\big)}{\vartheta_{\lambda+1}\big(\frac{\omega'}{\omega}\big)}\,\mathrm{e}^{\eta(\omega,\omega')\frac{z^2}{2\omega}}, \qquad \lambda = 1,2,3.$$

The Weierstrass σ-function, as function of $(z,\,g_2,\,g_3)$, satisfies linear differential equations obtained by Weierstrass himself:

$$z\frac{\partial\sigma}{\partial z} - 4g_2\frac{\partial\sigma}{\partial g_2} - 6g_3\frac{\partial\sigma}{\partial g_3} - \sigma = 0,$$

$$\frac{\partial^2\sigma}{\partial z^2} - 12g_3\frac{\partial\sigma}{\partial g_2} - \frac{2}{3}g_2^2\frac{\partial\sigma}{\partial g_3} + \frac{1}{12}g_2z^2\sigma = 0.$$

It immediately follows that there exists a recursive relation for coefficients C_k of the power series

$$\sigma(z;g_2,g_3) = C_0z + C_1\frac{z^3}{3!} + \cdots = z - \frac{g_2}{240}z^5 - \frac{g_3}{840}z^7 + \cdots, \tag{9}$$

where the standard normalization $\sigma(0) = 0$, $\sigma'(0) = 1$, $\sigma''(0) = 0$ has been adopted. The two classical recurrences are known. The first one is due to Halphen [40, **I**: p. 300]:

$$C_k = \widehat{\mathfrak{D}}\, C_{k\text{-}1} - \frac{1}{6}(k-1)(2k-1)\, g_2 C_{k\text{-}2}\,, \quad \text{where} \quad \widehat{\mathfrak{D}} := 12 g_3 \frac{\partial}{\partial g_2} + \frac{2}{3} g_2^2 \frac{\partial}{\partial g_3}\,, \tag{10}$$

but in different notation it was written down by Weierstrass [77, **V**: p. 49] and even by Jacobi (see Sect. 5). The second one was obtained by Weierstrass:

$$\sigma(z; g_2, g_3) = \sum_{m,n=0}^{\infty} A_{m,n} \left(\frac{g_2}{2}\right)^m (2g_3)^n \frac{z^{4m+6n+1}}{(4m+6n+1)!}\,, \tag{11}$$

$$\begin{aligned} A_{m,n} = & \frac{16}{3}(n+1)\, A_{m-2,n+1} + 3(m+1)\, A_{m+1,n-1} - \\ & - \frac{1}{3}(2m+3n-1)(4m+6n-1)\, A_{m-1,n}\,, \end{aligned}$$

where $A_{0,0} = 1$ and $A_{m,n} = 0$ under $n, m < 0$. Other recurrences are also known [28]. Among all the recurrences the Weierstrassian one is least expendable because it contains only multiplication of integers and coefficients C_k have already been collected in parameters. It is interesting to remark that Weierstrass proves separately the fact that $A_{m,n}$ have integral values [77, **V**: p. 50]. We shall be guided by the same motivation when deriving the power series for Jacobi's θ-functions in Sect. 3.

2.3. **Dedekind's function.** Since the standard notations for Weierstrassian η-function and Dedekind's one coincide, we use for the latter the symbol $\boldsymbol{\eta}(\tau)$:

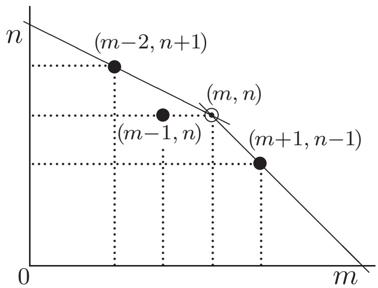
$$\boldsymbol{\eta}(\tau) = \mathrm{e}^{\frac{\pi \mathrm{i}}{12}\tau} \prod_{1}^{\infty}{}_k \left(1 - \mathrm{e}^{2k\pi \mathrm{i}\tau}\right) = \mathrm{e}^{\frac{\pi \mathrm{i}}{12}\tau} \sum_{-\infty}^{+\infty}{}_k (-1)^k\, \mathrm{e}^{(3k^2+k)\pi \mathrm{i}\tau} \qquad \big(\text{Euler (1748)}\big).$$

Dedekind's function is connected with the Jacobi–Weierstrass ones through the differential and algebraic relations [63, 75]:

$$\frac{1}{\boldsymbol{\eta}} \frac{d\boldsymbol{\eta}}{d\tau} = \frac{\mathrm{i}}{\pi}\eta, \qquad 2\boldsymbol{\eta}^3 = \vartheta_2 \vartheta_3 \vartheta_4\,. \tag{12}$$

3. Canonical power θ-series

Before proceeding to the θ-series we need some preparatory material on series for Weierstrassian σ's and graphical illustration to the recurrence $A_{m,n}$:



This picture means that computation of the point (m, n) involves computations of all the points inside this quadrangle.

Throughout the paper we use the symbol $[n]$ for integer part of the number n. Then one can easily show that Weierstrassian series (11) simplifies into the following expression explicitly collected in variable z:

$$\sigma(z;g_2,g_3) = \sum_{0}^{\infty}{}_{k} \left\{ \sum_{[k/3]}^{k/2}{}_{\nu} 2^{2k-5\nu} A_{3\nu-k,\,k-2\nu} \cdot g_2^{3\nu-k} g_3^{k-2\nu} \right\} \frac{z^{2k+1}}{(2k+1)!} \,. \tag{13}$$

Analogous series exist for all the σ-functions but their form depends on which parameters $g_{2,3}$ or e_λ are chosen as the basic ones (see Remark 1 further below).

3.1. **Halphen's operator and series for σ_λ.** Let us denote $e_\lambda := \wp(\omega_\lambda|\omega,\omega')$.

Lemma 1. *Power series for functions σ_λ are given by the expression*

$$\sigma_\lambda(z;e_\lambda,g_2) = \sum_{0}^{\infty}{}_{k} \left\{ \sum_{0}^{k/2}{}_{\nu}\, 2^{-\nu}\, \mathfrak{B}_{k-2\nu,\nu} \cdot e_\lambda^{k-2\nu} g_2^{\nu} \right\} \frac{z^{2k}}{(2k)!}$$

with the following integral recurrence:

$$\begin{aligned} \mathfrak{B}_{m,n} &= 24(n+1)\,\mathfrak{B}_{m-3,n+1} + (4m-12n-5)\,\mathfrak{B}_{m-1,n} \\ &\qquad - \frac{4}{3}(m+1)\,\mathfrak{B}_{m+1,n-1} - \frac{1}{3}(m+2n-1)(2m+4n-3)\,\mathfrak{B}_{m,n-1}, \\ \mathfrak{B}_{0,0} &= 1 \quad \textit{and} \quad \mathfrak{B}_{m,n} = 0 \ \ \textit{if} \ \ m,n<0. \end{aligned}$$

Proof. Calculations are based on Halphen's equations satisfied by σ_λ-functions [40]:

$$\begin{gathered} z\frac{\partial\sigma_\lambda}{\partial z} - 2e_\lambda\frac{\partial\sigma_\lambda}{\partial e_\lambda} - 4g_2\frac{\partial\sigma_\lambda}{\partial g_2} = 0, \\ \frac{\partial^2\sigma_\lambda}{\partial z^2} - \Big(4e_\lambda^2 - \frac{2}{3}g_2\Big)\frac{\partial\sigma_\lambda}{\partial e_\lambda} - 12\,(4e_\lambda^3 - g_2 e_\lambda)\frac{\partial\sigma_\lambda}{\partial g_2} + \Big(e_\lambda + \frac{1}{12}g_2 z^2\Big)\sigma_\lambda = 0 \end{gathered} \tag{14}$$

and proof of integrality of the recurrence $\mathfrak{B}_{m,n}$ is analogous to Weierstrassian argument on p. 50 in [77, **V**]. ∎

We can also include into this recurrence the σ-function. In this case analog of Halphen's equation (14) takes the form

$$\begin{gathered} z\frac{\partial\Xi}{\partial z} - 2e_\lambda\frac{\partial\Xi}{\partial e_\lambda} - 4g_2\frac{\partial\Xi}{\partial g_2} - (1-\varepsilon)\Xi = 0, \\ \frac{\partial^2\Xi}{\partial z^2} - \Big(4e_\lambda^2 - \frac{2}{3}g_2\Big)\frac{\partial\Xi}{\partial e_\lambda} - 12\,(4e_\lambda^3 - g_2 e_\lambda)\frac{\partial\Xi}{\partial g_2} + \Big(\varepsilon e_\lambda + \frac{1}{12}g_2 z^2\Big)\Xi = 0, \end{gathered} \tag{15}$$

where case $\Xi = \sigma_\lambda$ corresponds to $\varepsilon = 1$ and $\Xi = \sigma$ corresponds to $\varepsilon = 0$ with arbitrary e_λ. This is because σ-function does not depends on permutation of branch-points e_λ.

Corollary 2. *Power series for all the Weierstrass σ-functions are defined by the following expression*

$$\Xi(z;e_\lambda,g_2) = \sum_{0}^{\infty}{}_{k} \left\{ \sum_{0}^{k/2}{}_{\nu}\, 2^{-\nu}\, \mathfrak{B}^{(\varepsilon)}_{k-2\nu,\nu} \cdot e_\lambda^{k-2\nu} g_2^{\nu} \right\} \frac{z^{2k+1-\varepsilon}}{(2k+1-\varepsilon)!} \tag{16}$$

under the universal integral recurrence

$$\mathfrak{B}^{(\varepsilon)}_{m,n} = 24(n+1)\,\mathfrak{B}^{(\varepsilon)}_{m-3,n+1} + (4m-12n-4-\varepsilon)\,\mathfrak{B}^{(\varepsilon)}_{m-1,n} - \frac{4}{3}(m+1)\,\mathfrak{B}^{(\varepsilon)}_{m+1,n-1} - \frac{1}{3}(m+2n-1)(2m+4n-1-2\varepsilon)\,\mathfrak{B}^{(\varepsilon)}_{m,n-1}.$$

Remark 1. Weierstrass himself wrote out recurrences not for his σ-functions but for functions $S_\lambda = \exp\left\{\frac{1}{2}e_\lambda z^2\right\}\sigma_\lambda(z)$ with parameters $\left(e_\lambda, \varepsilon_\lambda = 3e_\lambda^2 - \frac{1}{4}g_2\right)$ [77, **II**: p. 253–254]. A possible reason is that functions S_λ yield a four-term recurrence like $A_{m,n}$, whereas our recurrences are five-term ones. However, one can show that recurrences for functions σ_λ through parameters (g_2, g_3) do not exist. Nevertheless transition between pairs (e_λ, g_2) and (e_λ, e_μ) is one-to-one and therefore this universal recurrence may be written out in any of these 'representations'. It will be completely symmetric in the (e_λ, e_μ)-representation.

Another kind of formulae for coefficients of the series is the ϑ-constant one. It is a natural choice for the power θ-series. Indeed, the ϑ-constant expressions for branch points e_λ are well known. In turn, ϑ-constants are related by the Jacobi identity $\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4$. This allows us to pass between the representations choosing arbitrary pair. If formulae are written through the constants $(\vartheta_{\alpha 0}, \vartheta_{0\beta})$, where $(\alpha,\beta) \neq (0,0)$, we shall use the term (α,β)*-representation.*

The foregoing Halphen equations and their other representations can be written in form of one equation if we avail of operator (10). We can derive, e. g., that

$$\widehat{\mathfrak{D}} = \left(4e_\lambda^2 - \frac{2}{3}g_2\right)\frac{\partial}{\partial e_\lambda} + 12\left(4e_\lambda^3 - g_2\,e_\lambda\right)\frac{\partial}{\partial g_2}$$

and symmetrical form of this operator reads

$$= \frac{4}{3}\left(e_\lambda^2 - 2e_\mu e_\lambda - 2e_\mu^2\right)\frac{\partial}{\partial e_\lambda} + \frac{4}{3}\left(e_\mu^2 - 2e_\lambda e_\mu - 2e_\lambda^2\right)\frac{\partial}{\partial e_\mu}.$$

The ϑ-constant form for $\widehat{\mathfrak{D}}$-operator is also easily calculated since all the parameters of the theory, i. e., g_2, g_3, e_λ, and $\vartheta_{\alpha\beta}$ are related to each other by the polynomial relations [69]. For example $(\vartheta_2, \vartheta_4)$-representation for $\widehat{\mathfrak{D}}$ reads

$$\widehat{\mathfrak{D}} = \frac{\pi^2}{3}\left(\vartheta_4^8 + 2\vartheta_2^4\vartheta_4^4\right)\frac{\partial}{\partial(\vartheta_4^4)} - \frac{\pi^2}{3}\left(\vartheta_2^8 + 2\vartheta_2^4\vartheta_4^4\right)\frac{\partial}{\partial(\vartheta_2^4)}\,.$$

Let us denote

$$\langle\alpha\rangle := (-1)^\alpha.$$

Then the general (α,β)-representation to the Halphen operator is given by the following formula:

$$\widehat{\mathfrak{D}} = \frac{\pi^2}{3}\left(\langle\beta\rangle\,\vartheta_{\alpha 0}^8 + 2\,\langle\alpha\rangle\,\vartheta_{\alpha 0}^4\,\vartheta_{0\beta}^4\right)\frac{\partial}{\partial(\vartheta_{\alpha 0}^4)} - \frac{\pi^2}{3}\left(\langle\alpha\rangle\,\vartheta_{0\beta}^8 + 2\,\langle\beta\rangle\,\vartheta_{\alpha 0}^4\,\vartheta_{0\beta}^4\right)\frac{\partial}{\partial(\vartheta_{0\beta}^4)}\,.$$

Define also the symbol ε depending on parity of the function $\theta_{\alpha\beta}$:

$$\varepsilon := \frac{\langle\alpha\beta\rangle + 1}{2} \quad \Rightarrow \quad \begin{cases} \varepsilon = 0 & \text{if} \;\; \theta_{\alpha\beta} = \pm\theta_1 \\ \varepsilon = 1 & \text{if} \;\; \theta_{\alpha\beta} = \pm\theta_{2,3,4} \end{cases}.$$

Lemma 3. *Halphen's equation* (15) *for the functions* $\Xi = \{\sigma, \sigma_\lambda\}$ *has the form*

$$\frac{\partial^2 \Xi}{\partial z^2} - \widehat{\mathfrak{D}}\,\Xi + \Big\{\varepsilon\, e_\lambda(\vartheta) + \frac{\pi^4}{12^2}\big(\vartheta_2^8 + \vartheta_2^4\vartheta_4^4 + \vartheta_4^8\big) z^2\Big\}\Xi = 0$$

and its general (α,β)*-representation is as follows*

$$\frac{\partial^2 \Xi}{\partial z^2} - \widehat{\mathfrak{D}}\,\Xi + \Big\{e_{\gamma\delta}(\vartheta) + \frac{\pi^4}{12^2}\Big[\vartheta_{\alpha0}^8 + \langle\alpha{+}\beta\rangle\,\vartheta_{\alpha0}^4\vartheta_{0\beta}^4 + \vartheta_{0\beta}^8\Big] z^2\Big\}\Xi = 0. \tag{17}$$

Here, quantities

$$e_{\gamma\delta}(\vartheta) = \frac{\pi^2}{12}\big(\langle\gamma\rangle\,\vartheta_{0\delta}^4 - \langle\delta\rangle\,\vartheta_{\gamma0}^4\big) \tag{18}$$

do not dependent on representation (α,β) *and correspond to functions* σ, σ_λ *by the rules*

$$\sigma \leftrightarrow e_{00} = 0, \qquad \sigma_1 \leftrightarrow e_{01} = e_1, \qquad \sigma_2 \leftrightarrow e_{11} = e_2, \qquad \sigma_3 \leftrightarrow e_{10} = e_3.$$

We say that representation is a *proper* one (or symmetric) if $(\gamma,\delta) = (\alpha,\beta)$.

Remark 2 (historical). Nice recurrences on the plane (analogs of $A_{m,n}$) were obtained by Weierstrass already in the 1840's. At the time he was using his old notation Al instead of Jacobi's Θ and H [77, **I**]. At about the same time Jacobi considered power series for his theta-functions and introduced the important multiplier e^{Az^2} much as Weierstrass did this for his σ-function. See further Remark 4 in Sect. 5 or lectures by Koenigsberger [50, **II**: p. 79–81].

3.2. **Power θ-series.** The functions θ are fundamental objects in numerous theories. For this reason we shall give representation for their power series in a maximally simplified (canonical) form. Because of this, we do collect all the parameters similarly to Weierstrassian recurrence $A_{m,n}$ so that only multiplications of integers remain. It is not difficult to see that the series under question must be series with coefficients being polynomials in variables $\eta(\tau)$, $\vartheta(\tau)$. This follows from the obvious formulae

$$\theta_1(z|\tau) = \pi\boldsymbol{\eta}^3(\tau)\cdot \mathrm{e}^{-2\eta(\tau)z^2}\sigma(2z|\tau)\,, \qquad \theta_\lambda(z|\tau) = \vartheta_\lambda(\tau)\cdot \mathrm{e}^{-2\eta(\tau)z^2}\sigma_{\lambda-1}(2z|\tau). \tag{19}$$

By this means all the formulae that follows are derivable with use of various ϑ-representations to operator $\widehat{\mathfrak{D}}$ in Halphen's equations (14), (15), and (17) followed by multiplying the result into the series for an exponent. Computations are somewhat lengthy but routine and we therefore omit them entirely.

Theorem 4. *Power series for the function*

$$\begin{aligned}\theta_1(z|\tau) &= \sum_{0}^{\infty}{}_k\, C_k(\tau)\cdot z^{2k+1}\\ &= 2\pi\boldsymbol{\eta}^3\Big\{z - 2\eta\cdot z^3 + \Big(2\eta^2 - \frac{\pi^4}{180}\big(\vartheta_2^8 + \vartheta_2^4\vartheta_4^4 + \vartheta_4^8\big)\Big)\cdot z^5 + \cdots\Big\}\end{aligned} \tag{20}$$

is determined by the following analytic expression

$$\begin{aligned}\theta_1(z|\tau) = {}&\quad 2\pi\sum_{0}^{\infty}{}_k\,\frac{(4\pi\mathrm{i})^k}{(2k+1)!}\,\frac{d^k\boldsymbol{\eta}^3}{d\tau^k}\cdot z^{2k+1}\\ &= 2\pi\boldsymbol{\eta}^3\sum_{0}^{\infty}{}_k\,(-2)^k\Big\{\sum_{0}^{k}{}_\nu\Big(-\frac{\pi^2}{6}\Big)^\nu\frac{\eta^{k-\nu}\boldsymbol{\mathcal{N}}_\nu(\vartheta)}{(k-\nu)!\,(2\nu+1)!}\Big\}z^{2k+1}\,,\end{aligned} \tag{21}$$

where ϑ-polynomial

$$\mathcal{N}_\nu(\vartheta) = \sum_0^\nu{}_s \left\{ \begin{array}{r} \mathfrak{G}_{\nu-s,s} \cdot \vartheta_4^{4s}\,\vartheta_2^{4(\nu-s)} \\ (-1)^s\,\mathfrak{G}_{\nu-s,s} \cdot \vartheta_3^{4s}\,\vartheta_4^{4(\nu-s)} \\ (-1)^s\,\mathfrak{G}_{s,\nu-s} \cdot \vartheta_3^{4s}\,\vartheta_2^{4(\nu-s)} \end{array} \right\}$$

is chosen (in braces) according to which ϑ-constant $(4,2)$-, $(3,4)$-, or $(3,2)$-representation is taken. Here, the integral recurrence $\mathfrak{G}_{m,n}$ is defined as follows

$$\begin{aligned} \mathfrak{G}_{m,n} &= 4(n-2m-1)\,\mathfrak{G}_{m,n-1} - 4(m-2n-1)\,\mathfrak{G}_{m-1,n} \\ &\qquad - 2(m+n-1)(2m+2n-1)\big(\mathfrak{G}_{m-2,n} + \mathfrak{G}_{m-1,n-1} + \mathfrak{G}_{m,n-2}\big), \\ \mathfrak{G}_{0,0} &= 1 \quad \textit{and} \quad \mathfrak{G}_{m,n} = 0 \ \ \textit{if} \ \ m,n<0 \end{aligned}$$

and has the symmetry property $\mathfrak{G}_{m,n} = (-1)^{m+n}\,\mathfrak{G}_{n,m}$.

Remark 3. We might of course derive representation of the type $\theta_1 = \sum C_{mnp}\, g_2^m\, g_3^n\, \eta^p\, z^k$ like Weierstrassian recurrence, but $\mathfrak{G}_{m,n}$ is more effective than $A_{m,n}$ since all the polynomials have already been collected in ϑ-constants.

It is interesting to observe that odd derivatives $\theta_1^{(2k+1)}(0|\tau)$, i. e., coefficients in front of z^{2k+1} in (20)–(21), generate polynomial expressions in variables (η,ϑ) which are exactly integrable k times in τ.

Theorem 5. *Power series for the functions $\theta[^\alpha_\beta] = \pm\theta_{2,3,4}$:*

$$\begin{aligned} \theta[^\alpha_\beta](z|\tau) &= \sum_0^\infty{}_k\, C_k^{(\alpha,\beta)}(\tau)\cdot z^{2k} = \\ &= \vartheta[^\alpha_\beta] - \vartheta[^\alpha_\beta]\Big\{2\eta + \frac{1}{6}\pi^2\Big(\langle\beta\rangle\,\vartheta[^{\alpha-1}_{\ 0}]^4 - \langle\alpha\rangle\,\vartheta[^{\ 0}_{\beta-1}]^4\Big)\Big\}z^2 + \cdots \end{aligned} \tag{22}$$

are determined by the following analytic expressions

$$\theta[^\alpha_\beta](z|\tau) = \sum_0^\infty{}_k\, \frac{(4\pi\mathrm{i})^k}{(2k)!}\,\frac{d^k\vartheta[^\alpha_\beta]}{d\tau^k}\cdot z^{2k}\,. \tag{23}$$

The proper representation to the series (23) has the form

$$\theta[^\alpha_\beta](z|\tau) = \vartheta[^\alpha_\beta]\sum_0^\infty{}_k\,(-2)^k\left\{\sum_0^k{}_\nu\left(-\frac{\pi^2}{6}\right)^\nu \frac{\eta^{k-\nu}\,\mathcal{N}_\nu^{(\alpha,\beta)}(\vartheta)}{(k-\nu)!\,(2\nu)!}\right\}z^{2k} \tag{24}$$

with the following universal integral recurrence:

$$\mathcal{N}_\nu^{(\alpha,\beta)}(\vartheta) = \sum_0^\nu{}_s\, \mathfrak{G}_{s,\nu-s}^{(\alpha,\beta)}\cdot\vartheta[^{\alpha-1}_{\ 0}]^{4s}\,\vartheta[^{\ 0}_{\beta-1}]^{4(\nu-s)}\,,$$

$$\begin{aligned} \mathfrak{G}_{m,n}^{(\alpha,\beta)} &= \langle\alpha\rangle(4n-8m-3)\,\mathfrak{G}_{m,n-1}^{(\alpha,\beta)} - \langle\beta\rangle(4m-8n-3)\,\mathfrak{G}_{m-1,n}^{(\alpha,\beta)} \\ &\qquad - 2(m+n-1)(2m+2n-3)\big(\mathfrak{G}_{m-2,n}^{(\alpha,\beta)} + \langle\alpha+\beta\rangle\,\mathfrak{G}_{m-1,n-1}^{(\alpha,\beta)} + \mathfrak{G}_{m,n-2}^{(\alpha,\beta)}\big), \end{aligned}$$

where $\mathfrak{G}_{0,0}^{(\alpha,\beta)} = 1$ and $\mathfrak{G}_{m,n}^{(\alpha,\beta)} = 0$ if $m,n<0$.

Some remarks are in order. These recurrences are quite effective but there are additional symmetry properties which reduce computations in half. It is evident from the recurrence $\mathfrak{G}_{m,n}^{(\alpha,\beta)}$ itself that it has a symmetry with respect to permutations of indices

$$\mathfrak{G}_{n,m}^{(\alpha,\beta)} = (-1)^{(m+n)(\alpha+\beta+1)}\,\mathfrak{G}_{m,n}^{(\alpha,\beta)}\,, \qquad \mathfrak{G}_{m,n}^{(\beta,\alpha)} = (-1)^{(m+n)(\alpha+\beta)}\,\mathfrak{G}_{m,n}^{(\alpha,\beta)}\,. \tag{25}$$

This means that we have in effect only two recurrences $(\alpha,\beta)=(1,0)$ and $(\alpha,\beta)=(0,0)$, i. e., β is always equal to zero. Redenoting $\mathfrak{G}_{m,n}^{(\alpha,\beta)} = \mathfrak{G}_{m,n}^{(\alpha,0)} =: \mathfrak{G}_{m,n}^{(\alpha)}$, we have

$$\begin{aligned}\mathfrak{G}_{m,n}^{(\alpha)} = \langle\alpha\rangle\,(4n-8m-3)\,\mathfrak{G}_{m,n-1}^{(\alpha)} - (4m-8n-3)\,\mathfrak{G}_{m-1,n}^{(\alpha)}\\ -2\,(m+n-1)(2m+2n-3)\big(\mathfrak{G}_{m-2,n}^{(\alpha)} + \langle\alpha\rangle\,\mathfrak{G}_{m-1,n-1}^{(\alpha)} + \mathfrak{G}_{m,n-2}^{(\alpha)}\big)\end{aligned}$$

and permutations (25) therefore reduce to the simple formulae

$$\mathfrak{G}_{n,m}^{(0)} = (-1)^{(m+n)}\,\mathfrak{G}_{m,n}^{(0)}\,, \qquad \mathfrak{G}_{n,m}^{(1)} = \mathfrak{G}_{m,n}^{(1)}\,.$$

We see that recurrences (21) and (24) differ only in multipliers. Hence they can be unified into one recurrence much as we did it in (16) by introducing the parity ε, but the quantity $\langle\alpha\rangle$ still remains. Computer tests show that (m,n)-entries of matrices $\mathfrak{G}^{(\beta,\alpha)}$ differ each other only in signs but we failed to find this rule.

Corollary 6. *All the coefficients $C_k(\tau)$ and $C_k^{(\alpha,\beta)}(\tau)$ are the k-fold exactly τ-integrable (η,ϑ)-polynomials.*

In Sect. 7 we shall show that this integrability is a consequence of one dynamical system. With use of the formulae above one can construct series in neighborhoods of points $z = \{\pm\frac{1}{2}, \pm\frac{\tau}{2}\}$. By virtue of (4), the resulting series are transformed into each other with some obvious modifications.

4. Dynamical systems satisfied by θ-series

In this and next section we describe new and important property of Jacobi's ϑ, θ, and θ'-series. These, along with elliptic, elementary, or rational functions, are differentially closed and define thereby the calculus in its own right.

Theorem 7. *The five functions $\theta_1(z|\tau)$, $\theta_2(z|\tau)$, $\theta_3(z|\tau)$, $\theta_4(z|\tau)$, and $\theta_1'(z|\tau)$ satisfy the closed autonomous ordinary differential equations over the field of coefficients ϑ_2, ϑ_3, ϑ_4, and η:*

$$\left\{\begin{aligned}
\frac{\partial\theta_1}{\partial z} &= \theta_1'\\
\frac{\partial\theta_2}{\partial z} &= \frac{\theta_1'}{\theta_1}\theta_2 - \pi\vartheta_2^2\cdot\frac{\theta_3\theta_4}{\theta_1}\\
\frac{\partial\theta_4}{\partial z} &= \frac{\theta_1'}{\theta_1}\theta_4 - \pi\vartheta_4^2\cdot\frac{\theta_2\theta_3}{\theta_1}\\
\frac{\partial\theta_3}{\partial z} &= \frac{\theta_1'}{\theta_1}\theta_3 - \pi\vartheta_3^2\cdot\frac{\theta_2\theta_4}{\theta_1}\\
\frac{\partial\theta_1'}{\partial z} &= \frac{\theta_1'^2}{\theta_1} - \pi^2\vartheta_3^2\vartheta_4^2\cdot\frac{\theta_2^2}{\theta_1} - 4\left\{\eta + \frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\right\}\cdot\theta_1
\end{aligned}\right. . \tag{26}$$

Proof. It is based on the theta-function differential identities which occur infrequently in the literature (mostly in the old books; [75, p. 82], [69, **II**: p. 173][3]). These relationships are nothing else but ones between Weierstrassian functions $(\sigma, \zeta)(z|\tau)$ taken at different half-periods [76]. We can present them in the following compact form:

$$\theta_\mu \theta_\nu' - \theta_\nu \theta_\mu' = \mathfrak{sign}(\nu - \mu)\pi \vartheta_k^2 \cdot \theta_1 \theta_k, \tag{27}$$

where $k = 2, 3, 4$ and

$$\nu = \frac{8k - 28}{3k - 10}, \qquad \mu = \frac{10k - 28}{3k - 8}; \tag{28}$$

the triple (k, ν, μ) runs over the set $\{(2,3,4), (3,4,2), (4,2,3)\}$. In order to turn identities (27) into differential equations we should find their differential closure. Taking the property $\vartheta_1 \equiv 0$ into account, we can solve (27) with respect to the θ-derivatives and rewrite the result as first four equations in (26):

$$\frac{\partial \theta_k}{\partial z} = \frac{\theta_1'}{\theta_1}\theta_k - \pi \vartheta_k^2 \cdot \frac{\theta_\nu \theta_\mu}{\theta_1}, \qquad k = 1, 2, 3, 4.$$

It only remains to compute derivative of the object θ_1'. Consider Weierstrassian identity

$$(\sigma\zeta)' = \sigma\zeta^2 - \sigma\wp$$

and convert it into the θ-functions. Then formulae

$$\zeta(2z|\tau) = 2\eta(\tau)z + \frac{1}{2}\frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)}, \tag{29}$$

$$\wp(2z|\tau) = \frac{\pi^2}{12}\left\{\vartheta_3^4(\tau) + \vartheta_4^4(\tau) + 3\vartheta_3^2(\tau)\vartheta_4^2(\tau)\frac{\theta_2^2(z|\tau)}{\theta_1^2(z|\tau)}\right\} \tag{30}$$

and formula (19) yield the fifth equation in (26). ∎

Weierstrass himself derived a (ζ, σ_λ)-equivalent of relations (27) in a reverse order [76, §§24–25], i. e., by differentiating the σ-identities followed by use of differential equations for ratios of σ-functions. It should be noted here that the closed θ-form of Weierstrassian identities contains not only branch points e_λ, i. e., ϑ-constants, but also the 'constant' η. In other words, the closed differential θ-apparatus inevitably contains a fifth function—any of θ_k'—and period of a meromorphic elliptic integral, i. e., $\eta(\tau)$; the total number of equations is thus equal to five.

As mentioned in Introduction, Theorem 7 appears explicable on the basis of the theory of Abelian integrals. Namely, these integrals are differentially closed and elliptic functions are the particular case of the meromorphic integrals (integrals of exact differentials). The logarithmic integral is a logarithmic θ-ratio and canonical meromorphic integral—Weierstrassian ζ-function—is proportional to the fifth function θ_1'.

It is notable that the famous Jacobi identity $\vartheta_1' = \pi\vartheta_2\vartheta_3\vartheta_4$ turns out to be an automatical consequence of Eqs. (26) taken at point $z = 0$ and this property pertains equally to generalizations of this identity presented by formulae (42)–(44) next. By this means we get one more (simple) proof of Jacobi's identity meanwhile in the Whittaker–Watson

[3]These important relations are very implicitly present in Jacobi's *Werke* [48, **I**] but have not got to the thorough handbook for elliptic functions [76] compiled by Schwarz from Weierstrassian lectures. Even lectures themselves [77] contain no these identities in the θ-form. They present in [76, p. 29], [77] in form of their (ζ, σ_λ)-equivalents. Differential relations for quotients of θ-functions are of course well known. These are differential equations for elliptic functions [79, Sect. 21·6], [8, 52, 50, 75].

book [79] all the known proofs of this identity are summarized as 'none are simple' [79, Sect. 21·41].

The next step suggests itself. All the θ, θ'-functions satisfy the heat equation:

$$4\pi\mathrm{i}\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial z^2}, \qquad 4\pi\mathrm{i}\frac{\partial\theta'}{\partial\tau} = \frac{\partial^2\theta'}{\partial z^2}. \tag{31}$$

Therefore, invoking Theorem 7, we establish that differential closedness is shared also by theta-functions as functions of their second argument.

Theorem 8. *The five Jacobi's functions $\theta_k(z|\tau)$ and $\theta_1'(z|\tau)$ satisfy the closed non-autonomous ordinary differential equations*:

$$\begin{aligned}
\frac{\partial\theta_1}{\partial\tau} &= \frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^2}{\theta_1} + \frac{\pi\mathrm{i}}{4}\vartheta_3^2\vartheta_4^2\cdot\frac{\theta_2^2}{\theta_1} + \frac{\mathrm{i}}{\pi}\Big\{\eta + \frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\Big\}\cdot\theta_1,\\
\frac{\partial\theta_2}{\partial\tau} &= \frac{-\mathrm{i}}{4\pi}\Big\{\frac{\theta_1'}{\theta_1} - \pi\vartheta_2^2\cdot\frac{\theta_3\theta_4}{\theta_1\theta_2}\Big\}^2\theta_2 + \frac{\pi\mathrm{i}}{4}\vartheta_3^2\vartheta_4^2\cdot\frac{\theta_1^2}{\theta_2} + \frac{\mathrm{i}}{\pi}\Big\{\eta + \frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\Big\}\cdot\theta_2,\\
\frac{\partial\theta_3}{\partial\tau} &= \frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^2}{\theta_1^2}\theta_3 + \frac{\mathrm{i}}{2}\vartheta_3^2\cdot\theta_1'\frac{\theta_2\theta_4}{\theta_1^2} - \frac{\pi\mathrm{i}}{4}\vartheta_2^2\vartheta_3^2\cdot\frac{\theta_4^2}{\theta_1^2}\theta_3 + \frac{\mathrm{i}}{\pi}\Big\{\eta + \frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\Big\}\cdot\theta_3,\\
\frac{\partial\theta_4}{\partial\tau} &= \frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^2}{\theta_1^2}\theta_4 + \frac{\mathrm{i}}{2}\vartheta_4^2\cdot\theta_1'\frac{\theta_2\theta_3}{\theta_1^2} - \frac{\pi\mathrm{i}}{4}\vartheta_2^2\vartheta_4^2\cdot\frac{\theta_3^2}{\theta_1^2}\theta_4 + \frac{\mathrm{i}}{\pi}\Big\{\eta + \frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\Big\}\cdot\theta_4,\\
\frac{\partial\theta_1'}{\partial\tau} &= \frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^3}{\theta_1^2} + \frac{3\mathrm{i}}{\pi}\Big\{\frac{\pi^2}{4}\vartheta_3^2\vartheta_4^2\cdot\frac{\theta_2^2}{\theta_1^2} + \eta + \frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\Big\}\theta_1' - \frac{\pi^2}{2}\mathrm{i}\vartheta_2^2\vartheta_3^2\vartheta_4^2\cdot\frac{\theta_2\theta_3\theta_4}{\theta_1^2}.
\end{aligned}$$

When deriving second of these equations the heat equation (31) was used. This system, combined with Eqs. (26), constitutes a complete set of rules for differential computations with theta-series and, invoking notation (28), they can be written down in the following compact form:

$$\left\{\begin{aligned}
\frac{\partial\theta_k}{\partial z} &= \frac{\theta_1'}{\theta_1}\theta_k - \pi\vartheta_k^2\cdot\frac{\theta_\nu\theta_\mu}{\theta_1}\\
\frac{\partial\theta_1'}{\partial z} &= \frac{\theta_1'^2}{\theta_1} - \pi^2\vartheta_3^2\vartheta_4^2\cdot\frac{\theta_2^2}{\theta_1} - 4\Big\{\eta + \frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\Big\}\cdot\theta_1
\end{aligned}\right., \tag{32}$$

$$\left\{\begin{aligned}
\frac{\partial\theta_k}{\partial\tau} &= \frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^2}{\theta_1^2}\theta_k + \frac{\mathrm{i}}{2}\vartheta_k^2\cdot\theta_1'\frac{\theta_\nu\theta_\mu}{\theta_1^2} + \frac{\pi\mathrm{i}}{4}\Big\{\vartheta_3^2\vartheta_4^2\cdot\theta_2^2 - \vartheta_k^2\vartheta_\mu^2\cdot\theta_\nu^2 - \vartheta_k^2\vartheta_\nu^2\cdot\theta_\mu^2\Big\}\frac{\theta_k}{\theta_1^2}\\
&\quad + \frac{\mathrm{i}}{\pi}\Big\{\eta + \frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\Big\}\cdot\theta_k\\
\frac{\partial\theta_1'}{\partial\tau} &= \frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^3}{\theta_1^2} + \frac{3\mathrm{i}}{\pi}\Big\{\frac{\pi^2}{4}\vartheta_3^2\vartheta_4^2\cdot\frac{\theta_2^2}{\theta_1^2} + \eta + \frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\Big\}\theta_1' - \frac{\pi^2}{2}\mathrm{i}\vartheta_2^2\vartheta_3^2\vartheta_4^2\cdot\frac{\theta_2\theta_3\theta_4}{\theta_1^2}
\end{aligned}\right., \tag{33}$$

where $k = 1, 2, 3, 4$. These formulae, incidentally, are not completely symmetric and heat equation was not involved when deriving them; this important point is discussed in Sect. 9.3.

5. ϑ-CONSTANT DIFFERENTIAL CALCULUS

Equations (33) contain $\vartheta(\tau)$- and $\eta(\tau)$-constants but their derivatives have not been defined yet. On the other hand, Weierstrass' invariants (6) have their ϑ-constant equivalents

$$\begin{aligned} g_2(\tau) &= \frac{\pi^4}{24}\{\vartheta_2^8(\tau)+\vartheta_3^8(\tau)+\vartheta_4^8(\tau)\}, \\ g_3(\tau) &= \frac{\pi^6}{432}\{\vartheta_2^4(\tau)+\vartheta_3^4(\tau)\}\{\vartheta_3^4(\tau)+\vartheta_4^4(\tau)\}\{\vartheta_4^4(\tau)-\vartheta_2^4(\tau)\} \end{aligned} \tag{34}$$

and satisfy the famous Halphen dynamical system [40, **I**: p. 331, 449–450]

$$\frac{dg_2}{d\tau}=\frac{\mathrm{i}}{\pi}\Big(8g_2\eta-12g_3\Big), \qquad \frac{dg_3}{d\tau}=\frac{\mathrm{i}}{\pi}\Big(12g_3\eta-\frac{2}{3}g_2^2\Big), \qquad \frac{d\eta}{d\tau}=\frac{\mathrm{i}}{\pi}\Big(2\eta^2-\frac{1}{6}g_2\Big), \tag{35}$$

involving the η-function. In implicit form this system was also written down by Weierstrass [77, **II**: p. 249] and Ramanujan obtained its equivalent [62] when studying his known number-theoretic P, Q, R-series. It immediately follows that differential closure of the ϑ's requires an extension of the system (35) to a 4-dimensional version. It is a direct corollary of Eqs. (34)–(35).

Theorem 9. *Jacobi's ϑ-constants differentially closed upon adjoining the Weierstrass η-function*:

$$\begin{aligned} \frac{d\vartheta_2}{d\tau} &= \frac{\mathrm{i}}{\pi}\left\{\eta+\frac{\pi^2}{12}\left(\vartheta_3^4+\vartheta_4^4\right)\right\}\vartheta_2, & \frac{d\vartheta_4}{d\tau} &= \frac{\mathrm{i}}{\pi}\left\{\eta-\frac{\pi^2}{12}\left(\vartheta_2^4+\vartheta_3^4\right)\right\}\vartheta_4, \\ \frac{d\vartheta_3}{d\tau} &= \frac{\mathrm{i}}{\pi}\left\{\eta+\frac{\pi^2}{12}\left(\vartheta_2^4-\vartheta_4^4\right)\right\}\vartheta_3, & \frac{d\eta}{d\tau} &= \frac{\mathrm{i}}{\pi}\left\{2\eta^2-\frac{\pi^4}{12^2}\left(\vartheta_2^8+\vartheta_3^8+\vartheta_4^8\right)\right\}. \end{aligned} \tag{36}$$

Remark 4 (historical). It is less known that Jacobi wrote out a different analog to this 4-dimensional dynamical system but Halphen does not mention this fact. This result was published by Borchardt in 1857 on the basis of manuscripts kept after Jacobi [48, **II**: p. 383–398]. Namely, Jacobi introduced the four variables (A, B, a, b) in terms of Legendre's quantities (k, k', K, E) and showed that they satisfy the nice monomial dynamical system (we keep completely to Jacobi's notation in [48, p. 386])

$$\left\{ \begin{aligned} \frac{\partial A}{\partial h} &= 2A^2B, & \frac{\partial a}{\partial h} &= -16bA^2, \\ \frac{\partial B}{\partial h} &= bA^3, & \frac{\partial b}{\partial h} &= abA^2, \end{aligned} \right. \tag{37}$$

where $h=\frac{1}{4}\pi\mathrm{i}\tau$. Interestingly enough, Jacobi considered system (37) in a context of the power series for θ-functions and noticed [48, **II**: p. 390] that the series would be simple if one extracts the exponential multiplier $\exp\{-\frac{1}{2}ABz^2\}$. He described corresponding recurrences for θ_k [48, **II**: p. 394–398] and one can readily see that they are equivalent to the Weierstrass–Halphen differential recurrence (10) for σ-functions. Exhaustive comments to the Jacobi system and its relation to Eqs. (36) can be found in [13].

Before the system (37) was derived, Jacobi also obtained its analogs (see [48, **II**: p. 176]) and, in particular, his remarkable differential equation of 3rd order for the ϑ-series

$$C^4\big(\ln C^3C_{\tau\tau}\big)_\tau^2 = 16C^3C_{\tau\tau}-\pi^2, \qquad C=\vartheta^{-2}. \tag{38}$$

(C is Jacobi's notation). In turn, a simple computation shows that logarithmic derivatives of the ϑ-series also satisfy a compact differential equation which we shall meet in Sect. 9.3. The equation and its general solution are as follows:

$$\big(X_\tau-2X^2\big)X_{\tau\tau\tau}-X_{\tau\tau}^2+16X^3X_{\tau\tau}+4\big(X_\tau-6X^2\big)X_\tau^2=0, \tag{39}$$

$$X=\frac{d}{d\tau}\ln\frac{\vartheta_k\Big(\dfrac{a\tau+b}{c\tau+d}\Big)}{\sqrt{c\tau+d}}.$$

We should also mention that well-known differential relations on logarithms of ratios $\vartheta_2:\vartheta_3:\vartheta_4$ [75, 69]

$$\frac{d}{d\tau}\ln\frac{\vartheta_2}{\vartheta_3}=\frac{\pi}{4}\mathrm{i}\vartheta_4^4,\qquad \frac{d}{d\tau}\ln\frac{\vartheta_3}{\vartheta_4}=\frac{\pi}{4}\mathrm{i}\vartheta_2^4,\qquad \frac{d}{d\tau}\ln\frac{\vartheta_2}{\vartheta_4}=\frac{\pi}{4}\mathrm{i}\vartheta_3^4$$

(see also old dissertation [8]) and ordinary differential equation of Chazy [23, 24]

$$\pi\eta_{\tau\tau\tau}=12\mathrm{i}\big(2\eta\eta_{\tau\tau}-3\eta_\tau^2\big) \tag{40}$$

are the direct consequences of the system (36).

Remark 5 (exercise). If we view the last equation in (35) as a Riccati equation then we get an interesting example of the solvable linear 2nd order ODE. Coefficient of this equation is proportional to everywhere holomorphic in $\mathbb{H}^+$ form $g_2(\tau)$ which is an automorphic one with respect to group $\mathbf{\Gamma}(1)$. Carry out the calculations and bring the equation into the following form

$$\Psi''+\frac{g_2(\tau)}{3\pi^2}\Psi=0.$$

With use of differential relation (12) generalize this equation to equation

$$\Psi''+\frac{n+2}{\pi\mathrm{i}}\eta(\tau)\Psi'-\frac{n}{6\pi^2}g_2(\tau)\Psi=0$$

and prove that

$$\Psi=\frac{1}{\boldsymbol{\eta}^n(\tau)}\Big(A+B\int^\tau\boldsymbol{\eta}^{2n}(\tau)\,d\tau\Big)$$

is its general solution.

In conclusion of this section, we note that formulae for multiple differentiating the ϑ,η-constants are given explicitly as coefficients of series (23)–(24). The same coefficients provide the general expressions for quantities $\theta^{(n)}(0|\tau)$; relations between derivatives $\theta'(0|\tau)$, $\theta''(0|\tau),\ldots,\theta^{(n)}(0|\tau)$ under small n are very often used in the literature as auxiliary identities [57, 79, 32, 69, 75, 52, 50, 41, 65, 54] (Baruch's dissertation [8] contains a lot of such identities). See also work [17] where modular functions like η, ϑ, ϑ', etc appear in the theory of hydrodynamical chains and differential calculus described above significantly simplifies computations in this work. Thanks to the fact that group $\mathbf{\Gamma}(1)$ has a lot of interesting and nontrivial subgroups the number of known differential systems related to the base one (36) is far from being exhausted. Even next to $\mathbf{\Gamma}(1)$ groups like $\mathbf{\Gamma}_0(N)$ lead a rich theory. See, for example, recent work [56] containing many nice results along these lines and additional bibliography.

6. Unification: θ, θ'-functions with characteristics

In this section we summarize and represent the previous results and other basic properties of theta-functions and their derivatives in a unified notation, i. e., in terms of theta-characteristics with use of (α, β)-representations. This will enable us primarily to trivialize and automate analytic manipulation with theta-functions by including fundamental operations: shifts by half-periods, modular transformations, and differential computations. Apart from unification of formulae this can serve as the basis for further generalization to the theta-functions of higher genera.

Any object, symmetrical in ϑ-constants, can be written in (α, β)-representation. For example branch points (18) or (α, β)-representation for invariants (34) can be written as follows:

$$g_2(\tau) = \frac{\pi^4}{12}\left\{\vartheta_{\alpha 0}^8 + \langle \alpha+\beta\rangle\, \vartheta_{\alpha 0}^4 \vartheta_{0\beta}^4 + \vartheta_{0\beta}^8\right\}, \qquad (\alpha,\beta) \neq (0,0)$$

$$g_3(\tau) = \frac{\pi^6}{432}\left\{2\langle\beta\rangle\, \vartheta_{\alpha 0}^{12} - 3\vartheta_{\alpha 0}^4 \vartheta_{0\beta}^4 \left(\langle\beta\rangle\, \vartheta_{0\beta}^4 - \langle\alpha\rangle\, \vartheta_{\alpha 0}^4\right) - 2\langle\alpha\rangle\, \vartheta_{0\beta}^{12}\right\}.$$

Other examples are the Jacobi identity

$$\vartheta_3^4(\tau) = \vartheta_2^4(\tau) + \vartheta_4^4(\tau) \tag{41}$$

and formula $\vartheta_1' = 2\pi\boldsymbol{\eta}^3$; they have the following (α, β)-representation:

$$\vartheta[^{\alpha}_{\beta}]^4 = \left(\langle\beta\rangle\, \vartheta[^{\alpha-1}_{0}]^4 + \langle\alpha\rangle\, \vartheta[^{0}_{\beta-1}]^4\right)\frac{\langle\alpha\beta\rangle + 1}{2}, \qquad \vartheta_{\alpha\beta}'(\tau) = \mathrm{i}^{\beta+1}\left(1 - \langle\alpha\beta\rangle\right)\cdot \pi\boldsymbol{\eta}^3(\tau). \tag{42}$$

Here and hereafter $\vartheta_{\alpha\beta}'(\tau)$ is understood to be equal to $\theta_{\alpha\beta}'(0|\tau)$.

6.1. **Shifts by half-periods for θ-derivatives.** In connection with appearance of the object θ_1', we should augment the rule (4) by involving the fact that algebraic and differential closedness of θ's entails a transformation law for their derivatives. Hence, it is naturally to be expected that any function $\theta_{\alpha\beta}'(z|\tau)$ with a z-argument shifted by some half-period is expressible through the function $\theta_1'(z|\tau)$ and other functions $\theta_{1,2,3,4}(z|\tau)$. The ultimate answer is, however, not a simple differential consequence of formula (4) and is far from being obvious. It should be put to better use as an independent property.

Theorem 10 (Transformation law for θ-derivatives)**.** *Let $\alpha, \beta, m, n \in \mathbb{Z}$. Then*

$$\theta_{\alpha\beta}'\left(z + \frac{n}{2} + \frac{m}{2}\tau \middle| \tau\right) = \mathrm{i}^{3m(\beta+n)}\cdot \mathrm{e}^{-\frac{1}{4}\pi \mathrm{i} m(4z+m\tau)}\Big\{\left(\theta_1'(z|\tau) - \pi\mathrm{i} m\,\theta_1(z|\tau)\right)\theta[^{\alpha+m}_{\beta+n}](z|\tau)$$
$$- \left\langle(\alpha+m)\left[\tfrac{\beta+n}{2}\right]\right\rangle \pi\,\vartheta[^{\alpha+m}_{\beta+n}]^2\cdot \theta[^{\alpha+m-1}_{0}](z|\tau)\,\theta[^{0}_{\beta+n-1}](z|\tau)\Big\}\frac{1}{\theta_1(z|\tau)},$$

where, for closedness of the formula, identity (3) should be taken into account.

Proof. It is a combination of equations (32) and conversion of any $\theta_{\alpha\beta}$-function into the function θ_1 by formula

$$\theta_1(z|\tau) = \mathrm{i}^{\alpha}\,\theta[^{\alpha-1}_{\beta-1}]\left(z - \frac{\alpha}{2}\tau - \frac{\beta}{2}\middle|\tau\right)\cdot \mathrm{e}^{-\pi\mathrm{i}\alpha\left(z - \frac{1}{4}\alpha\tau\right)}, \tag{43}$$

wherein we set (α, β) to be integers (m, n). ∎

Taking the limit at $z = 0$, which exploits the series expansions described above, we get a generalization of Jacobi's derivative formula (second formula in (42)).

Corollary 11. *The general $\theta'_{\alpha\beta}$-constant, i. e., value of θ'-function at any half-period, is expressed through a ϑ-constant and exponential multiplier*:

$$\theta'_{\alpha\beta}\Big(\frac{n}{2}+\frac{m}{2}\tau\Big|\tau\Big) = \mathrm{i}^{1-(\beta+n)m}\cdot\pi\Big\{\mathrm{i}^{\beta+n}\big(1-\langle(\alpha{+}m)(\beta{+}n)\rangle\big)\cdot\boldsymbol{\eta}^3 - m\,\vartheta\big[\begin{smallmatrix}\alpha+m\\ \beta+n\end{smallmatrix}\big]\Big\}\,\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}m^2\tau}. \quad (44)$$

Since $\langle(\alpha{+}m)(\beta{+}n)\rangle$ is equal to ±1, only one term remains in the right-hand side of (44), i. e., $\vartheta\big[\begin{smallmatrix}\alpha+m\\ \beta+n\end{smallmatrix}\big]$ or $\boldsymbol{\eta}^3 = \frac{1}{2}\vartheta_2\vartheta_3\vartheta_4$. For tables of some particular examples see [69, **II**: p. 256].

6.2. **Modular transformations.** Transformations of θ-functions with respect to modular group $\big(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\big) \in \boldsymbol{\Gamma}(1)$ belong among fundamental properties of theta-functions and have numerous applications [63, 6]. Suffice it to mention that corresponding transformation of the series θ_1 [78, 69, 75]

$$\theta_1\Big(\frac{z}{c\tau+d}\Big|\frac{a\tau+b}{c\tau+d}\Big) = \aleph^3\cdot\sqrt{c\tau+d}\;\mathrm{e}^{\frac{\pi\mathrm{i}cz^2}{c\tau+d}}\,\theta_1(z|\tau), \tag{45}$$

where $\aleph^3$ denotes some eighth root of unity, may turn a hyper-convergent series into the never computable one. Hermite represented the famous multiplier $\aleph^3$ via the sum of quadratic Gaussian exponents (it is known that these sums are not easily computed) and Jacobi's symbol $\big(\frac{a}{b}\big)$ [44, **I**: p. 482–486] (see also [53, p. 183–193], [50, **II**: p. 57–58], [75, p. 124–132], [69, **II**]). For this reason, it is interesting that there is a simpler formula for the modular transformation wherein multiplier $\aleph$ is merely an exponent of a rational. Without loss of generality we may normalize c to be positive: $c > 0$.

Theorem 12 (The $\boldsymbol{\Gamma}(1)$-transformation law for the general θ-function)**.** *Let $\theta\big[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\big]$ be the theta-series with arbitrary integer characteristics* (2) *and $n\in\mathbb{Z}$. Then*

$$\theta\big[\begin{smallmatrix}\alpha-1\\ \beta\end{smallmatrix}\big](z|\tau+n) = \mathrm{i}^{\frac{1}{2}n(1-\alpha^2)}\cdot\theta\big[\begin{smallmatrix}\alpha-1\\ \beta+n\alpha\end{smallmatrix}\big](z|\tau), \tag{46}$$

$$\theta\big[\begin{smallmatrix}\alpha'-1\\ \beta'-1\end{smallmatrix}\big]\Big(\frac{z}{c\tau+d}\Big|\frac{a\tau+b}{c\tau+d}\Big) = \mathfrak{E}_{\alpha\beta}\,\aleph^3\cdot\sqrt{c\tau+d}\;\mathrm{e}^{\frac{\pi\mathrm{i}cz^2}{c\tau+d}}\,\theta\big[\begin{smallmatrix}\alpha-1\\ \beta-1\end{smallmatrix}\big](z|\tau), \tag{47}$$

where multipliers $\mathfrak{E}_{\alpha\beta}$ and $\aleph$ depend on (a,b,c,d) as follows

$$\mathfrak{E}_{\alpha\beta} = \exp\frac{\pi}{4}\mathrm{i}\big\{2\alpha(\beta bc-d+1)-\beta c(\beta a-2)-\alpha^2db\big\},$$

$$\aleph = \exp\pi\mathrm{i}\left\{\frac{a-d}{12c}-\frac{d}{6}(2c-3)+\frac{c-1}{4}\,\mathfrak{sign}(d)-\frac{1}{4}+\frac{1}{c}\cdot\sum_{\|c/d\|+1}^{c-1}{}_{k}\Big[\frac{d}{c}k\Big]\,k\right\} \tag{48}$$

and characteristics (α,β), (α',β') are related through the linear transformation

$$\begin{aligned}\alpha' &= d\alpha-c\beta, &\qquad \alpha &= a\alpha'+c\beta',\\ \beta' &= -b\alpha+a\beta, &\qquad \beta &= b\alpha'+d\beta'.\end{aligned} \tag{49}$$

Proof. Formula (46) is an elementary consequence of the series (2). Proof of (47) consists of two steps. The first one is accurate manipulations/simplifications by Dedekind's sums [61, 6] determining the multiplier $\aleph$ and entering into the transformation formula for the $\boldsymbol{\eta}$-function:

$$\boldsymbol{\eta}\Big(\frac{a\tau+b}{c\tau+d}\Big) = \aleph\sqrt{c\tau+d}\,\boldsymbol{\eta}(\tau).$$

Ensuing use of the fact that multiplier for θ_1 in (45) is a cube of multiplier $\aleph$ for $\boldsymbol{\eta}$ [69, 78] yields formula (48). The second step exploits the fact that function θ_1 transforms into itself and any of the functions $\theta_{\alpha\beta}(z|\tau)$ can be transformed into the function $\theta_1(z|\tau)$ (and vice versa) by a half-period shift of its z-argument like (43). This gives the linear transformation between characteristics (49), and with it the multiplier $\mathfrak{E}_{\alpha\beta}$. Characteristics, as appeared in (47), have been chosen in order that the formula be most symmetric. ■

Remark 6. Hermite gave also *nonlinear* formulae for transformation of characteristics $(\alpha,\beta)\mapsto(\alpha',\beta')$ [44, **I**: p. 483] which are reproduced in subsequent works [75, 33, 64] (the linear form can be found in [53, p. 183]). Somewhat surprising facet is the fact that no such a self-contained formula seems to have hitherto been presented in the literature. Ratio of any θ-functions contains no the multiplier $\aleph$ and Hermite used this fact to build the functions $\varphi(\tau)$, $\psi(\tau)$, $\chi(\tau)$ and tables of transformation between them [44, 69] when constructing his famous solution to the quintic equation $x^5-x=a$ in terms of φ, ψ, χ [44, p. 10]. These functions are in fact certain ϑ-constants so that their transformations are consequences of the ϑ-constant ones.

Corollary 13. *The* $\mathbf{\Gamma}(1)$*-transformations for the general* $\vartheta\big[{\alpha \atop \beta}\big]$*-constants are*

$$\vartheta\big[{\alpha-1 \atop \beta}\big](\tau+n)=\mathrm{i}^{\frac{n}{2}(1-\alpha^2)}\cdot\vartheta\big[{\alpha-1 \atop \beta+n\alpha}\big](\tau),$$

$$\vartheta\big[{\alpha'-1 \atop \beta'-1}\big]\Big(\frac{a\tau+b}{c\tau+d}\Big)=\mathrm{e}^{\frac14\pi\mathrm{i}\{2\alpha(\beta bc-d+1)-\beta c(\beta a-2)-\alpha^2 db\}}\,\aleph^3\cdot\sqrt{c\tau+d}\,\vartheta\big[{\alpha-1 \atop \beta-1}\big](\tau).$$

The known property for each θ_k-function to transforms into itself under the group $\mathbf{\Gamma}(2)$ is also consequence of Theorem 12 and formulae (49).

Corollary 14. *Let* (m,n,p,q) *be integers. Then group* $\mathbf{\Gamma}(2)\ni\big({2n+1 \atop 2p}\ {2m \atop 2q+1}\big)=\big({a \atop c}\ {b \atop d}\big)$ *is necessary and sufficient for each function* θ_k *to transform into itself*:

$$\theta_1\Big(\frac{z}{c\tau+d}\Big|\frac{a\tau+b}{c\tau+d}\Big)=\aleph^3\cdot\sqrt{c\tau+d}\ \mathrm{e}^{\frac{\pi\mathrm{i}cz^2}{c\tau+d}}\,\theta_1(z|\tau),$$

$$\theta_2\Big(\frac{z}{c\tau+d}\Big|\frac{a\tau+b}{c\tau+d}\Big)=\mathrm{i}^{2q(p-1)+p}\,\aleph^3\cdot\sqrt{c\tau+d}\ \mathrm{e}^{\frac{\pi\mathrm{i}cz^2}{c\tau+d}}\,\theta_2(z|\tau),$$

$$\theta_3\Big(\frac{z}{c\tau+d}\Big|\frac{a\tau+b}{c\tau+d}\Big)=\mathrm{i}^{2q(p+1)-m(2n+1)+p}\,\aleph^3\cdot\sqrt{c\tau+d}\ \mathrm{e}^{\frac{\pi\mathrm{i}cz^2}{c\tau+d}}\,\theta_3(z|\tau),$$

$$\theta_4\Big(\frac{z}{c\tau+d}\Big|\frac{a\tau+b}{c\tau+d}\Big)=\mathrm{i}^{2n(m-1)-m}\,\aleph^3\cdot\sqrt{c\tau+d}\ \mathrm{e}^{\frac{\pi\mathrm{i}cz^2}{c\tau+d}}\,\theta_4(z|\tau).$$

Proof. With use of (49) and (47) we get

$$\theta_2\Big(\frac{z}{c\tau+d}\Big|\frac{a\tau+b}{c\tau+d}\Big)=\mathrm{e}^{\frac14\pi\mathrm{i}(2-d)c}\,\aleph^3\cdot\sqrt{c\tau+d}\ \mathrm{e}^{\frac{\pi\mathrm{i}cz^2}{c\tau+d}}\ \theta\big[{c-1 \atop d-1}\big](z|\tau),$$

$$\theta_3\Big(\frac{z}{c\tau+d}\Big|\frac{a\tau+b}{c\tau+d}\Big)=\mathrm{e}^{\frac14\pi\mathrm{i}\{2(a+c-ad)-ab-cd\}}\,\aleph^3\cdot\sqrt{c\tau+d}\ \mathrm{e}^{\frac{\pi\mathrm{i}cz^2}{c\tau+d}}\,\theta\big[{a+c-1 \atop b+d-1}\big](z|\tau),$$

$$\theta_4\Big(\frac{z}{c\tau+d}\Big|\frac{a\tau+b}{c\tau+d}\Big)=\mathrm{e}^{\frac14\pi\mathrm{i}(2a-ab-2)}\,\aleph^3\cdot\sqrt{c\tau+d}\ \mathrm{e}^{\frac{\pi\mathrm{i}cz^2}{c\tau+d}}\ \theta\big[{a-1 \atop b-1}\big](z|\tau).$$

Requiring now $\theta\bigl[\begin{smallmatrix}c-1\\d-1\end{smallmatrix}\bigr]\simeq\theta\bigl[\begin{smallmatrix}1\\0\end{smallmatrix}\bigr]=\theta_2$, $\theta\bigl[\begin{smallmatrix}a+c-1\\b+d-1\end{smallmatrix}\bigr]\simeq\theta\bigl[\begin{smallmatrix}0\\0\end{smallmatrix}\bigr]=\theta_3$, and $\theta\bigl[\begin{smallmatrix}a-1\\b-1\end{smallmatrix}\bigr]\simeq\theta\bigl[\begin{smallmatrix}0\\1\end{smallmatrix}\bigr]=\theta_4$, we obtain *linear* equations for a,b,c,d. Their solution yields the matrix $\bigl(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\bigr)=\bigl(\begin{smallmatrix}2n+1&2m\\2p&2q+1\end{smallmatrix}\bigr)$. ■

Transformation for the basic function θ_1' follows from a derivative of (45):

$$\theta_1'\Bigl(\frac{z}{c\tau+d}\Bigl|\frac{a\tau+b}{c\tau+d}\Bigr)=\aleph^3\cdot\sqrt{c\tau+d}\;\mathrm{e}^{\frac{\pi\mathrm{i}cz^2}{c\tau+d}}\Bigl\{(c\tau+d)\theta_1'(z|\tau)+2\pi\mathrm{i}\,c\,z\,\theta_1(z|\tau)\Bigr\}.$$

(Exercise: with use of Theorem 10 derive the $\boldsymbol{\Gamma}(1)$-transformation law for the general $\theta_{\alpha\beta}'$-function.) It is not difficult to see that general transformation can always be brought to the form $\theta_k\mapsto\theta_k$ if we involve the inhomogeneous transformations of argument $z\mapsto\frac{z+s\tau+r}{c\tau+d}$.

6.3. **Differential equations.** The next theorem describes completely differential calculus of the classical Jacobi ϑ,θ,θ'-series in both variables z and τ.

Theorem 15. *Jacobi's $\theta_{\alpha\beta}$, $\theta'(z|\tau)$-series (2) and (5) with arbitrary integer characteristics (α,β), as functions of variables z and τ, are differentially closed over the field of $\eta(\tau)$- and $\vartheta^2(\tau)$-constants. Corresponding rules for differentiating are defined by the (α,β)-representation of z-equations (32)*

$$\left\{\begin{aligned}\frac{\partial\theta\bigl[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\bigr]}{\partial z}&=\frac{\theta_1'}{\theta_1}\theta\bigl[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\bigr]-(-1)^{\left[\frac{\beta}{2}\right]\alpha}\pi\,\vartheta\bigl[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\bigr]^2\cdot\frac{\theta\bigl[\begin{smallmatrix}\alpha-1\\0\end{smallmatrix}\bigr]\theta\bigl[\begin{smallmatrix}0\\\beta-1\end{smallmatrix}\bigr]}{\theta_1}\\\frac{\partial\theta_1'}{\partial z}&=\frac{\theta_1'^2}{\theta_1}-\pi^2\vartheta_3^2\vartheta_4^2\cdot\frac{\theta_2^2}{\theta_1}-4\Bigl\{\eta+\frac{\pi^2}{12}\bigl(\vartheta_3^4+\vartheta_4^4\bigr)\Bigr\}\cdot\theta_1\end{aligned}\right. \tag{50}$$

and by the (α,β)-equivalent of τ-equations (33)

$$\left\{\begin{aligned}\frac{\partial\theta\bigl[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\bigr]}{\partial\tau}&=\frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^2}{\theta_1^2}\theta\bigl[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\bigr]+\frac{\mathrm{i}}{2}(-1)^{\left[\frac{\beta}{2}\right]\alpha}\vartheta\bigl[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\bigr]^2\cdot\theta_1'\frac{\theta\bigl[\begin{smallmatrix}\alpha-1\\0\end{smallmatrix}\bigr]\theta\bigl[\begin{smallmatrix}0\\\beta-1\end{smallmatrix}\bigr]}{\theta_1^2}\\&\quad+\frac{\pi\mathrm{i}}{4}\Bigl\{\vartheta_3^2\vartheta_4^2\cdot\theta_2^2-\Bigl(\vartheta\bigl[\begin{smallmatrix}0\\\beta-1\end{smallmatrix}\bigr]^2\cdot\theta\bigl[\begin{smallmatrix}\alpha-1\\0\end{smallmatrix}\bigr]^2+\vartheta\bigl[\begin{smallmatrix}\alpha-1\\0\end{smallmatrix}\bigr]^2\cdot\theta\bigl[\begin{smallmatrix}0\\\beta-1\end{smallmatrix}\bigr]^2\Bigr)\vartheta\bigl[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\bigr]^2\Bigr\}\\&\quad+\frac{\mathrm{i}}{\pi}\Bigl\{\eta+\frac{\pi^2}{12}\bigl(\vartheta_3^4+\vartheta_4^4\bigr)\Bigr\}\cdot\theta\bigl[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\bigr]\frac{\theta\bigl[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\bigr]}{\theta_1^2}\\\frac{\partial\theta_1'}{\partial\tau}&=\frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^3}{\theta_1^2}+\frac{3\mathrm{i}}{\pi}\Bigl\{\frac{\pi^2}{4}\vartheta_3^2\vartheta_4^2\cdot\frac{\theta_2^2}{\theta_1^2}+\eta+\frac{\pi^2}{12}\bigl(\vartheta_3^4+\vartheta_4^4\bigr)\Bigr\}\theta_1'-\frac{\pi^2}{2}\mathrm{i}\,\vartheta_2^2\vartheta_3^2\vartheta_4^2\cdot\frac{\theta_2\theta_3\theta_4}{\theta_1^2}\end{aligned}\right.. \tag{51}$$

The constants $\eta(\tau)$, $\vartheta^2(\tau)$ form a differential ring $\mathbb{C}_\partial[\eta,\vartheta^2]$ that is defined by the following system of polynomial ODEs:

$$\left\{\begin{aligned}\frac{d\vartheta\bigl[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\bigr]}{d\tau}&=\frac{\mathrm{i}}{\pi}\Bigl\{\eta+\frac{\pi^2}{12}\Bigl((-1)^\beta\vartheta\bigl[\begin{smallmatrix}\alpha-1\\0\end{smallmatrix}\bigr]^4-(-1)^\alpha\vartheta\bigl[\begin{smallmatrix}0\\\beta-1\end{smallmatrix}\bigr]^4\Bigr)\Bigr\}\vartheta\bigl[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\bigr]\\\frac{d\eta}{d\tau}&=\frac{\mathrm{i}}{\pi}\Bigl\{2\eta^2-\frac{\pi^4}{72}\Bigl(\vartheta\bigl[\begin{smallmatrix}\alpha\\0\end{smallmatrix}\bigr]^8+(-1)^{\alpha+\beta}\vartheta\bigl[\begin{smallmatrix}\alpha\\0\end{smallmatrix}\bigr]^4\vartheta\bigl[\begin{smallmatrix}0\\\beta\end{smallmatrix}\bigr]^4+\vartheta\bigl[\begin{smallmatrix}0\\\beta\end{smallmatrix}\bigr]^8\Bigr)\Bigr\}\;\leftarrow\;(\alpha,\beta)\neq(0,0)\end{aligned}\right.. \tag{52}$$

We comment now on some connections of these dynamical systems with known properties of the theta-series. There are two fundamental algebraic relations between θ-series

$$\vartheta_2^2\theta_4^2-\vartheta_4^2\theta_2^2=\vartheta_3^2\theta_1^2,\qquad\vartheta_2^2\theta_3^2-\vartheta_3^2\theta_2^2=\vartheta_4^2\theta_1^2 \tag{53}$$

and they are of course compatible with equations (50) and (51) (proof is a calculation). However these relations are satisfied not only by θ-series themselves but by solutions of the equations as well. As before, a simple calculation shows that corresponding solutions contain three constants A, B, C and have the form

$$\begin{aligned} \theta[{}^{\alpha}_{\beta}] &= C\,e^{\pi \mathrm{i} A(2z+A\tau)}\cdot\theta[{}^{\alpha}_{\beta}](z+A\tau+B|\tau),\\ \theta_1' &= C\,e^{\pi \mathrm{i} A(2z+A\tau)}\cdot\Big\{\theta_1'(z+A\tau+B|\tau)-2\pi\mathrm{i}A\,\theta[{}^{1}_{1}](z+A\tau+B|\tau)\Big\}. \end{aligned} \tag{54}$$

Another point that should be noticed is the important fact that the heat equation (31) must be treated as a *corollary* of the above equations, rather than the reverse, because Eqs. (50)–(51) are the *ordinary* differential equations, while (31) is an equation in *partial* derivatives. The heat equation has a lot of solutions having nothing to do with theta-functions. In order to extract some special, say θ-, solutions to this equation we must impose additional (periodic, differential, modular, etc) conditions on them. Once this has been done for θ-functions, we arrive at the ODEs above, so that the initial consideration with the heat equation may be dropped out or 'forgotten'. In other words, equations (50)–(51) should be put to better consider as fundamental differential properties of theta-functions at all.

The third point we would like to mention here is the fact that modular transformation considered in Sect. 6.2 may be derived as a consequence of these equations rather than as an 'internal property' of theta-series themselves. This can be briefly outlined as follows. Equations (50)–(51) admit automorphisms which can be found through some linear fractional ansatz. That this ansatz is a linear fractional one can be determined with use of Lie's symmetries of these equations. This does not even use the fact that solutions to Eqs. (50)–(51) are the θ-series. Moreover, the availability of a discrete automorphism follows from the property that each function θ_k satisfies a *common ODE* (see Sect. 9.1). An analogous property holds for the ϑ-constant equation (38). By this means one can find two basic transformations $\tau\mapsto\tau+1$ and $\tau\mapsto -1/\tau$ generating group $\mathbf{\Gamma}(1)$. For lack of space we omit proofs of these statements but they can be partially compensated from the procedure of integration of the equations which shall be detailed in Sect. 9.4. Good examples of Lie's symmetries application to Chazy's equation (40) and many other Jacobi's 'modular/elliptic' equations are presented in nice works [24] and [66]. As with solutions (53) and (54) we can write solutions to system (52) that respect Jacobi's identity (41). These are separate solutions to Eqs. (38) and (40) and they are of course known:

$$\begin{aligned} \vartheta[{}^{\alpha}_{\beta}] &= \frac{1}{\sqrt{c\tau+d}}\cdot\vartheta[{}^{\alpha}_{\beta}]\Big(\frac{a\tau+b}{c\tau+d}\Big), &&\text{(Jacobi [48, \textbf{II}: p. 186–187])}\\ \eta &= \frac{1}{(c\tau+d)^2}\cdot\eta\Big(\frac{a\tau+b}{c\tau+d}\Big)+\frac{1}{2}\frac{\pi\mathrm{i}c}{c\tau+d}, &&\text{(Chazy [23])} \end{aligned} \tag{55}$$

where (a,b,c,d) are the integration constants, $ad-bc=1$, and right-hand sides in these formulae are the ϑ,η-series.

7. Compatible integrability of Eqs. (50)–(51)

An important corollary of the preceding section is that ODEs satisfied by Jacobi's functions have a wider class of solutions that are not bound to be the canonical θ-series (2) or formulae (54). The latter contain only three free constants while equations (50) are of fifth order.

In applications, variations of system (50) may occur in their own rights and hence the quantities ϑ's, being parameters in Eqs. (50), are not bound to be the values of θ-series at zero. For the same reason the quantity η must not necessarily be given by any known expression related to the ϑ, θ-series, e. g. [76, 69],

$$\eta(\tau) = -\frac{1}{12}\frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)} = -\frac{1}{4}\frac{\theta_3''(0|\tau)}{\vartheta_3(\tau)} - \frac{\pi^2}{12}\big\{\vartheta_2^4(\tau) - \vartheta_4^4(\tau)\big\} = \cdots .$$

Thus equations (50)–(51) may serve as an independent origin of the ϑ, θ-functions at all since equations are no less fundamental objects than their solutions.

7.1. **The θ-identities as algebraic integrals.** Let us assume that η and ϑ are the undetermined quantities in equations (50)–(51).

Theorem 16. *Equations (50)–(51) are compatible if and only if their coefficients η, ϑ_k satisfy the dynamical system*

$$\left\{\begin{aligned} \frac{d\vartheta_2}{d\tau} &= \frac{\mathrm{i}}{\pi}\left\{\eta + \frac{\pi^2}{12}\big(\vartheta_3^4 + \vartheta_4^4\big)\right\}\vartheta_2 \\ \frac{d\vartheta_3}{d\tau} &= \frac{\mathrm{i}}{\pi}\left\{\eta + \frac{\pi^2}{12}\big(\vartheta_3^4 + \vartheta_4^4 - 3\boldsymbol{B}^4\vartheta_4^4\big)\right\}\vartheta_3 \\ \frac{d\vartheta_4}{d\tau} &= \frac{\mathrm{i}}{\pi}\left\{\eta + \frac{\pi^2}{12}\big(\vartheta_3^4 + \vartheta_4^4 - 3\boldsymbol{A}^4\vartheta_3^4\big)\right\}\vartheta_4 \\ \frac{d\eta}{d\tau} &= \frac{\mathrm{i}}{\pi}2\eta^2 - \frac{\pi^3}{72}\mathrm{i}\Big\{\vartheta_3^8 + \big(9\boldsymbol{A}^4\boldsymbol{B}^4 - 6\boldsymbol{A}^4 - 6\boldsymbol{B}^4 + 2\big)\vartheta_3^4\vartheta_4^4 + \vartheta_4^8\Big\} \end{aligned}\right. , \tag{56}$$

where $\boldsymbol{A}^4$ and $\boldsymbol{B}^4$ are the algebraic (rational) integrals of the systems (50)–(51):

$$\boldsymbol{A}^4\cdot\vartheta_3^2\theta_1^2 = \vartheta_2^2\theta_4^2 - \vartheta_4^2\theta_2^2, \qquad \boldsymbol{B}^4\cdot\vartheta_4^2\theta_1^2 = \vartheta_2^2\theta_3^2 - \vartheta_3^2\theta_2^2. \tag{57}$$

The three functions $(\vartheta_3, \vartheta_4, \eta)$ are differentially closed.

Proof. Considering compatibility condition $\theta_{z\tau} = \theta_{\tau z}$ of systems (50) and (51), we obtain not only certain restrictions on coefficients η, ϑ but also algebraic relations between θ's. The straightforward check of (57) shows that $\boldsymbol{A}, \boldsymbol{B}(\theta;\vartheta)$ are the arbitrary constants indeed:

$$\frac{\partial\boldsymbol{A}}{\partial z} = \frac{\partial\boldsymbol{B}}{\partial z} \equiv 0, \qquad \frac{\partial\boldsymbol{A}}{\partial\tau} = \frac{\partial\boldsymbol{B}}{\partial\tau} \equiv 0$$

so that relations (57) do determine two independent *algebraic* integrals. ∎

Relations (57) generalize the well-known quadratic identities between canonical θ-series

$$\mathfrak{sign}(\nu-\mu)\cdot\vartheta_k^2\theta_1^2 = \vartheta_\mu^2\theta_\nu^2 - \vartheta_\nu^2\theta_\mu^2 \qquad (k = 2, 3, 4) \tag{58}$$

only two of which, say (53), are independent ones [75, 57] since ϑ-constants satisfy the Jacobi identity (41). It suggests that there exists one more integral; this is so indeed.

Theorem 17. *The system (56) has the algebraic (rational) integral $\mathfrak{A}^4(\vartheta)$:*

$$\mathfrak{A}^4\vartheta_2^4 = \boldsymbol{A}^4\vartheta_3^4 - \boldsymbol{B}^4\vartheta_4^4 \qquad \Rightarrow \qquad \frac{d}{d\tau}\mathfrak{A} \equiv 0. \tag{59}$$

Integrals (57) and (59), as generalizations of the famous Jacobi relations (58) and (41), mean that the various polynomial θ-identities, e. g., (58), are the additional constraints to the basic equations (50)–(51). We do not dwell on degenerations of system (56) into elementary functions and consider only generic situation describing non-canonical version of θ-functions. The quantities $\boldsymbol{A}$, $\boldsymbol{B}$ define initial conditions to Eqs. (50)–(51) and are parameters for system (56).

7.2. **Canonical θ-series and elliptic functions.** Let us explain how reduction to the canonical case of Jacobi–Weierstrass is performed. This procedure discloses some interesting facts.

Put $\boldsymbol{A} = \boldsymbol{B} = 1$. Then, as it follows from (56), function η satisfies the Chazy equation (40) and functions $\vartheta_{3,4}$ do the Jacobi equation (38). Integral (59) still remains to be free. Putting further $\mathfrak{A} = 1$, we can rewrite equations/identities (56) into the symmetrical form (36). This procedure, however, changes not only the structure of equations but their algebraic integral as well.

Proposition 18. *Algebraic integral $\mathfrak{A}(\vartheta)$ of canonical equations (36) has the form*

$$(\mathfrak{A}^4 - 1)\cdot\vartheta_2^4\vartheta_3^4\vartheta_4^4 = (\vartheta_3^4 - \vartheta_2^4 - \vartheta_4^4)^3. \tag{60}$$

Remark 7. This identity should be treated as a correct form of the 'complete Jacobi identity' if determining ODEs for quantities ϑ, η have been used in the symmetrized form (36). It is particularly remarkable that if we consider algebraic integrals (59) and (60) as algebraic curves in projective coordinates $\vartheta_2 : \vartheta_3 : \vartheta_4$. Then we find that curve (59) has genus three while (60) is a curve of genus nineteen! In addition to this complication under $\mathfrak{A} \ne 1$, none of the functions η, $\vartheta_{2,3,4}$, or logarithmic derivatives $\ln_\tau\vartheta$ satisfies any equation of 3rd order, as it was for equations (38)–(39). These assertions can be proved with use of polynomial Gröbner bases techniques over variables $\eta, \vartheta, \dot\eta, \dot\vartheta, \ldots$ [26] but we omit complete proofs for reasons of space. Broadly speaking, we loose a differential closedness of the three functions ϑ_3, ϑ_4, and η.

Remark 8. From the aforesaid it appears that we must do away with the rules of differentiations computations in the symmetrical form (36), (52) and redefine them according to Eqs. (56) under $\boldsymbol{A} = \boldsymbol{B} = 1$. Some heuristic arguments lead to the following formulae. The η-derivative becomes

$$\frac{d\eta}{d\tau} = \frac{\mathrm{i}}{\pi}\left\{2\eta^2 - \frac{\pi^4}{72}\left(\vartheta_3^8 - \vartheta_3^4\vartheta_4^4 + \vartheta_4^8\right)\right\}$$

and ϑ-constants are differentiated as

$$\frac{d\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}{d\tau} = \frac{\mathrm{i}}{\pi}\left\{\eta + \frac{\pi^2}{24}\Big[\big(2 - 3\langle\alpha\rangle - 3\langle\beta\rangle\big)\,\vartheta_4^4 - \big(1 - 3\langle\beta\rangle\big)\,\vartheta_3^4\Big]\right\}\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$$

or, equivalently,

$$\frac{d\vartheta_k}{d\tau} = \frac{\mathrm{i}}{\pi}\left\{\eta + \frac{\pi^2}{24}\Big[2\,(3k^2 - 18k + 25)\,\vartheta_4^4 - (3k^2 - 15k + 16)\,\vartheta_3^4\Big]\right\}\vartheta_k$$

under $k = 1, 2, 3, 4$ (recall $\vartheta_1 \equiv 0$) and, as before, arbitrary integral (α, β).

Let us denote $\mathbf{P} := \theta_2^2/\theta_1^2$. Then we find that the following equation holds:

$$\mathbf{P}_z^2 = 4\pi^2 \big(\vartheta_4^2 \cdot \mathbf{P} + \boldsymbol{A}^4 \vartheta_3^2\big) \big(\vartheta_3^2 \cdot \mathbf{P} + \boldsymbol{B}^4 \vartheta_4^2\big) \mathbf{P}. \tag{61}$$

Therefore $\mathbf{P}$ is expressible in terms of Weierstrass' $\wp$-function which is proportional to the ratio of Jacobi's θ-series by formula (30):

$$\wp(2z|\boldsymbol{\tau}) = \frac{\pi^2}{12}\left\{\vartheta_3^4(\boldsymbol{\tau}) + \vartheta_4^4(\boldsymbol{\tau}) + 3\vartheta_3^2(\boldsymbol{\tau})\vartheta_4^2(\boldsymbol{\tau})\,\frac{\theta_2^2(z|\boldsymbol{\tau})}{\theta_1^2(z|\boldsymbol{\tau})}\right\}.$$

From this point on we shall use notation $\vartheta_{2,3,4}(\boldsymbol{\tau})$, $\theta(z|\boldsymbol{\tau})$ for explicit pointing out the canonical ϑ, θ-series, their modulus $\boldsymbol{\tau}$, and argument z. The same symbols without arguments will denote dynamical variables entering into our ODEs. Bringing equation (61) into the form of Weierstrass' cubic and applying the standard technique [41, 79], we obtain that elliptic modulus $\boldsymbol{\tau}$ for equation (61) is determined as a root of the following transcendental equation (we recall (7))

$$J(\boldsymbol{\tau}) = \frac{1}{54}\,\frac{\big(\mathfrak{A}^8\vartheta_2^8 + \boldsymbol{A}^8\vartheta_3^8 + \boldsymbol{B}^8\vartheta_4^8\big)^3}{\mathfrak{A}^8\boldsymbol{A}^8\boldsymbol{B}^8 \cdot \vartheta_2^8\vartheta_3^8\vartheta_4^8}. \tag{62}$$

Here, for symmetry, we have used integral (59). Assuming for the moment that $\boldsymbol{\tau}$ has been determined, one derives that the following formula for the ratio $\mathbf{P}$ must take place:

$$\frac{\theta_2^2}{\theta_1^2} = \frac{\vartheta_3^4(\boldsymbol{\tau}) + \vartheta_4^4(\boldsymbol{\tau})}{3\vartheta_3^2\vartheta_4^2} - \frac{\boldsymbol{A}^4\vartheta_3^4 + \boldsymbol{B}^4\vartheta_4^4}{3\vartheta_3^2\vartheta_4^2} + \frac{\vartheta_3^2(\boldsymbol{\tau})\vartheta_4^2(\boldsymbol{\tau})}{\vartheta_3^2\vartheta_4^2} \cdot \frac{\theta_2^2(z+z_0|\boldsymbol{\tau})}{\theta_1^2(z+z_0|\boldsymbol{\tau})}.$$

Analogous formulae can be obtained for quotients θ_k/θ_j without squares. It is not difficult to see that such a variation would lead to the Jacobi elliptic functions $\mathrm{sn} \sim \theta_1/\theta_4$, etc:

$$\left(\frac{\theta_1}{\theta_4}\right)_z^2 = \pi^2 \left\{\boldsymbol{A}^4\vartheta_3^2 \cdot \left(\frac{\theta_1}{\theta_4}\right)^2 - \vartheta_2^2\right\} \left\{\mathfrak{A}^4\vartheta_2^2 \cdot \left(\frac{\theta_1}{\theta_4}\right)^2 - \vartheta_3^2\right\}. \tag{63}$$

We thus infer that ratio of any two θ-solutions to Eqs. (50)–(51) is proportional to a ratio of canonical θ-series with new modulus $\boldsymbol{\tau}$ determined from Eq. (62). Before proceeding to further integration, we make a digression about solution to that equations.

8. Modular inversion problem

As mentioned in Introduction, no explicit formula realization of the scheme (7) is hitherto available if elliptic curve has been given in Weierstrassian form

$$y^2 = 4x^3 - ax - b \tag{64}$$

or in more general form

$$y^2 = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4. \tag{65}$$

Analytic solution to the problem is known only for the canonical Legendre form

$$y^2 = (1 - x^2)(1 - k^2x^2). \tag{66}$$

In this case it is given by the famous formula of Jacobi

$$\tau = \mathrm{i}\,\frac{K'(k)}{K(k)}, \tag{67}$$

where K and K' are complete elliptic integrals [48, 32]. By virtue of classical formula

$$k^2 = \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}$$

this solution implies the identity

$$\tau \equiv \mathrm{i}\,\frac{K'\Bigl(\frac{\vartheta_2^2(\tau)}{\vartheta_3^2(\tau)}\Bigr)}{K\Bigl(\frac{\vartheta_2^2(\tau)}{\vartheta_3^2(\tau)}\Bigr)} \mod \boldsymbol{\Gamma}(2) \qquad \forall \tau \in \mathbb{H}^+. \tag{68}$$

Elliptic curves are, however, parametrized by group $\boldsymbol{\Gamma}(1)$, not by $\boldsymbol{\Gamma}(2)$. Moreover, transitions between (64), (65), and (66) requires knowledge of roots of the x-polynomials (65) or (64) (see [32, 13.5]) and modulus τ is computed via ratios of certain hypergeometric series. Owing to the fact that ${}_2F_1(J)$-series converges only inside the unity circle, the ${}_2F_1$-solutions will differ in structure depending on whether $|J| > 1$ or $|J| < 1$. For instance, in Weierstrassian representation (64) the resulting formulae constitute rather cumbersome expressions and, in addition to that, they involve the series in logarithmic derivative of Euler's Γ-function. See, e. g., one-half-page-long collection of formulae (22)–(27) in Sect. 14.6.2 of book [32] or enumeration of all the particular cases in [40, **I**: p. 341–348]. Such forms of solutions are not convenient in applications since they can not be manipulated analytically. Meanwhile the problem has an elegant solution.

8.1. **Analytic formula solution.** Modulus τ depends only on the value of absolute **J**-invariant which in turn is computed via coefficients a's through the two invariants [41, 5, 75]

$$g_2 = a_0 a_4 - 4a_1 a_3 + 3a_2^2, \qquad g_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} \tag{69}$$

according to Klein's definition

$$\mathbf{J} = \frac{g_2^3}{g_2^3 - 27\,g_3^2}. \tag{70}$$

It is well known that function $J(\tau)$ is related to a hypergeometric equation of the form [22]

$$J(J-1)\,\psi'' + \frac{1}{6}(7J-4)\,\psi' + \frac{1}{144}\psi = 0 \tag{71}$$

and generic solutions of such equations are usually designated by symbols ${}_2F_1(\alpha,\beta;\gamma|z)$. However, under some restrictions on parameters (α,β,γ) the solutions are representable in terms of known special functions; this point occurs when the ${}_2F_1$-series admits a quadratic transformation [31]. In this case ${}_2F_1$-equation reduces to an equation with two parameters, e. g., to Legendre's equation [4, 79, 31]. This is just the case of the ψ-equation above. Its solution is a linear combination of Legendre's functions

$$\psi = \sqrt[6]{J}\,\bigl\{A\,P_\nu^\mu(\sqrt{1-J}) + B\,Q_\nu^\mu(\sqrt{1-J})\bigr\}$$

with parameters $(\nu,\mu) = \bigl(-\frac{1}{2}, \frac{1}{3}\bigr)$. Recall that functions $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ are independent solutions of the following linear equation

$$(1-z^2)\,\psi'' - 2z\,\psi' + \Bigl\{\nu(\nu+1) - \frac{\mu^2}{1-z^2}\Bigr\}\psi = 0. \tag{72}$$

See [31, Ch. 3] for definitions and exhaustive properties of these functions. From the aforesaid it appears that the following formula must hold

$$\tau=\frac{\boldsymbol{a}\,P_\nu^\mu(\sqrt{1-J})+\boldsymbol{b}\,Q_\nu^\mu(\sqrt{1-J})}{\boldsymbol{c}\,P_\nu^\mu(\sqrt{1-J})+\boldsymbol{d}\,Q_\nu^\mu(\sqrt{1-J})},$$

where parameters $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})$ have definite numeric values. In order to find them, we need only any three values τ under which the quantity $J(\tau)$ has known exact values. There are lot of such points and all of them correspond to tori with complex multiplication [75]. For instance

$$J(\mathrm{i})=1,\qquad J\Big(\frac12+\mathrm{i}\frac12\sqrt3\Big)=0,\qquad J\big(\sqrt2\mathrm{i}\big)=\frac{5^3}{3^3},\quad \text{etc.}$$

The asymptotic property $P(z)/Q(z)\to\infty$ as $z\to\infty$ implies that $\boldsymbol{c}=0$ since $J(\mathrm{i}\infty)=\infty$. It therefore suffice to consider only two simplest points $J=\{0,1\}$ and corresponding values of functions $P,Q(\sqrt{1-J})$ are easily computed. We obtain these values with use of formulae (9)–(10) and table cases (22), (40) in Sect. 3.2 of [31].

Theorem 19 (Weierstrassian analog of (67))**.** *For elliptic curve in Weierstrassian form* (64) *its modulus $\tau=\omega'/\omega$, i. e., solution of the transcendental equation*

$$J(\tau)=\frac{a^3}{a^3-27b^2},$$

reads as follows

$$\tau=\mathrm{i}\,\frac{P^{\,0}_{-1/6}\big(-\sqrt{\mathfrak{g}}\big)}{P^{\,0}_{-1/6}\big(\sqrt{\mathfrak{g}}\big)},\qquad \mathfrak{g}:=27\,\frac{b^2}{a^3}. \tag{73}$$

If curve has the generic Legendre–Jacobi form (65) *then $\mathfrak{g}=1-\mathbf{J}^{-1}$ is computed according to* (69)–(70).

Proof. Let us use relations between Legendrian equations (72) with different indices. More precisely, carry out the above mentioned quadratic transformation

$$z\mapsto J=\frac{a\,z^2+b}{c\,z^2+d}$$

and demand preserving the normal form to Eq. (71), that is

$$\tilde\psi''=-\frac{1}{12^2}\,\frac{36\,J^2-41\,J+32}{(J-1)^2J^2}\,\tilde\psi.$$

We then get only the following possibilities:

$$z^2=1-J,\quad (\nu,\mu)=\Big(-\frac12,\,\pm\frac13\Big)$$

and

$$z^2=1-\frac1J,\quad (\nu,\mu)=\Big(-\frac16,\,0\Big)\quad\text{or}\quad(\nu,\mu)=\Big(-\frac56,\,0\Big).$$

Coefficients $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})$ are derived as said above. For the first case we obtain

$$\tau=\bigg\{\pi\,\mathrm{i}\,\frac{P_\nu^\mu}{Q_\nu^\mu}(\sqrt{1-J})-1\bigg\}\,e^{\frac{\pi}{3}\mathrm{i}}, \tag{74}$$

where $(\nu,\mu)=\big(-\frac12,\frac13\big)$. A simpler version comes from the second cases, e.g., from case $(\nu,\mu)=\big(-\frac16,0\big)$. Functions (P,Q) therewith are interchanged. As before, their proportionality coefficient is derived with use of tables in Sect. 3.2 of [31]. The last step is to use

formula 3.2(10) of [31] expressing a certain linear combination of $P(z)$ and $Q(z)$ through a single $P(-z)$; see also [4, (8.2.3)]. Under our parameters this relation reads

$$\pi P^{0}_{-1/6}(-z) = \pi\,\mathrm{e}^{\frac{\pi}{6}\mathrm{i}} P^{0}_{-1/6}(z) + Q^{0}_{-1/6}(z)$$

and simplifies the answer into the final formula (73). ■

The result (74) was announced recently in [12] and used there in connection with a nontrivial application to the soliton theory when considering a linear spectral problem of the form $\Psi''' + u(x)\,\Psi' - \frac{1}{2}u'(x)\,\Psi = \lambda\Psi$.

The function $\sqrt{1-J(\tau)}$ is a single-valued one and therefore we arrive at interesting analog of identity (68).

Corollary 20. *For all $\tau \in \mathbb{H}^+$ the following $\boldsymbol{\Gamma}(1)$-analog of identity (68) holds:*

$$\tau \equiv \left\{ \pi\,\mathrm{i}\,\frac{P^{\mu}_{\nu}}{Q^{\mu}_{\nu}}\Bigl(\mathrm{i}\frac{\sqrt{27}}{\pi^6}\frac{g_3(\tau)}{\boldsymbol{\eta}^{12}(\tau)}\Bigr) - 1 \right\} \mathrm{e}^{\frac{\pi}{3}\mathrm{i}} \mod \boldsymbol{\Gamma}(1)$$

under $(\nu,\mu) = \bigl(-\frac{1}{2}, \frac{1}{3}\bigr)$.

8.2. **Consequences.** An interrelation needs to be understood between (73) and Jacobi's formula (67) if elliptic curve (65) has already been given in the canonical form (66):

$$y^2 = (1-x^2)(1-\varkappa^2 x^2)\,.$$

Its absolute J-invariant is determined not by the classical expression

$$J = \frac{4}{27}\,\frac{(k^4-k^2+1)^3}{k^4\,(k^2-1)^2}\,, \tag{75}$$

wherein $k^2 = \varkappa^2$, but by expression of the form

$$J = \frac{1}{108}\,\frac{(\varkappa^4+14\,\varkappa^2+1)^3}{\varkappa^2\,(\varkappa^2-1)^4}\,. \tag{76}$$

In the standard Legendre–Jacobi theory of equation (66) [79, 69, 75, 40, 76] the function x is proportional to Jacobi's sn, whereas Weierstrass' $\wp$ is proportional to the square of sn; hence $\wp \rightleftarrows \mathrm{sn}^2$ is not a *birational* transformation. Function x also solves equation (63) under $\boldsymbol{A} = \mathfrak{A} = 1$ and hence has periods $2\mathrm{i}K'$ and $4K$, so that their ratio is equal to $\frac{1}{2}\tau$ rather than τ. To be more precise, one can carry out some standard calculations and derive birational transformations between Weierstrass' $\{\wp, \wp'\}$ and the θ-ratios, i. e., Jacobi's basis

$$\mathrm{sn}\bigl(\pi\vartheta_3^2(\tau)\,z;k\bigr) = \frac{\vartheta_3(\tau)}{\vartheta_2(\tau)}\cdot\frac{\theta_1(z|\tau)}{\theta_4(z|\tau)}\,,$$

$$\mathrm{cn}\bigl(\pi\vartheta_3^2(\tau)\,z;k\bigr) = \frac{\vartheta_4(\tau)}{\vartheta_2(\tau)}\cdot\frac{\theta_2(z|\tau)}{\theta_4(z|\tau)}\,,$$

$$\mathrm{dn}\bigl(\pi\vartheta_3^2(\tau)\,z;k\bigr) = \frac{\vartheta_4(\tau)}{\vartheta_3(\tau)}\cdot\frac{\theta_3(z|\tau)}{\theta_4(z|\tau)}\,.$$

Proposition 21 (Inversion of (30))**.** *Every homogeneous $\theta(z|\tau)$-ratio and consequently Jacobi's* $\{\mathrm{sn}, \mathrm{cn}, \mathrm{dn}\}$ *are rationally represented via Weierstrass' functions. The three basic θ-ratios read*

$$\begin{aligned}
\frac{\theta_2(z|\tau)}{\theta_1(z|\tau)} &= -\frac{\vartheta_2(\tau)}{\vartheta_1'(\tau)}\cdot\frac{\wp'(2z|2\tau)}{\wp(2z|2\tau)-e'(2\tau)}\\
&= 2\,\frac{\vartheta_2(\tau)}{\vartheta_1'(\tau)}\cdot\Big\{\zeta(2z|2\tau)-\zeta(2z-2\tau|2\tau)-\eta'(2\tau)\Big\},\\
\frac{\theta_3(z|\tau)}{\theta_1(z|\tau)} &= -\frac{\vartheta_3(\tau)}{2\,\vartheta_1'(\tau)}\cdot\frac{\wp'\big(z\big|\frac{\tau+1}{2}\big)}{\wp\big(z\big|\frac{\tau+1}{2}\big)-e\big(\frac{\tau+1}{2}\big)}\\
&= \frac{\vartheta_3(\tau)}{\vartheta_1'(\tau)}\cdot\Big\{\zeta\big(z\big|\tfrac{\tau+1}{2}\big)-\zeta\big(z-1\big|\tfrac{\tau+1}{2}\big)-\eta\big(\tfrac{\tau+1}{2}\big)\Big\},\\
\frac{\theta_4(z|\tau)}{\theta_1(z|\tau)} &= -\frac{\vartheta_4(\tau)}{2\,\vartheta_1'(\tau)}\cdot\frac{\wp'\big(z\big|\frac{\tau}{2}\big)}{\wp\big(z\big|\frac{\tau}{2}\big)-e\big(\frac{\tau}{2}\big)}\\
&= \frac{\vartheta_4(\tau)}{\vartheta_1'(\tau)}\cdot\Big\{\zeta\big(z\big|\tfrac{\tau}{2}\big)-\zeta\big(z-1\big|\tfrac{\tau}{2}\big)-\eta\big(\tfrac{\tau}{2}\big)\Big\}
\end{aligned}$$

and nine other ones are obtained by the half-period shifts $z \mapsto z + \frac{1}{2}\{1, \tau, \tau+1\}$.

Half-moduli on right hand sides of these equations explain the 'distinction' between Weierstrass' modulus for $(\zeta, \wp, \wp')$ and Jacobi's one for sn. By this means transition $\varkappa^2 \rightleftarrows k^2$, i. e., transformation (75)$\rightleftarrows$(76), is realized through a duplication of modulus:

$$\varkappa^2 = k^2(2\tau).$$

To put it differently, the map $\tau \mapsto 2\tau$ is a one-to-one transformation—it is just a normalization of $\tau \in \mathbb{H}^+$—and every elliptic curve is uniquely determined by the value τ (or 2τ). We might of course work with modulus 2τ instead of τ, however, in this case, classical integral representation of fundamental group $\mathbf{\Gamma}(1)$ must be changed to the matrices $\big(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\big)\mathbf{\Gamma}(1)\big(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\big)^{-1}$. As far as we know, no this precise correlation between Weierstrass' and Jacobi's modular inversions seems to have been mentioned in the literature [79, 30, 77, 76, 69, 75, 41, 5, 54, 32, 40].

One further comment is in order. Transformation of the curve (65) from general form into the Weierstrass one (and vice versa) is performed through the linear fractional change of variable x [22, 75, 41, 32]. This requires knowledge of roots of x-polynomial (65), i. e., solution of quartic equation, however. This is not convenient in investigation if coefficients of the polynomial do not have definite numerical values but are parameters. For this reason it would be useful to have a transformation over the field of coefficients $\mathbb{C}(a_0, \ldots, a_4)$, i. e., *without resorting to solution of any equations.* That birational change does exist indeed and below is its version. To simplify formulae we make a trivial transformation bringing (65) to a shorten form with $(a_0, a_1) = (1, 0)$.

Proposition 22. *The elliptic curve*

$$y^2 = x^4 - 6\alpha x^2 + 4\beta x + \gamma \tag{77}$$

is equivalent to the canonical Weierstrass form through a birational transformation over $\mathbb{C}(\alpha,\beta)$ *(no* γ *here). Corresponding Weierstrass' cubic has the form*

$$\boldsymbol{w}^2 = 4\boldsymbol{z}^3 - (3\alpha^2+\gamma)\,\boldsymbol{z} - (\alpha^3-\gamma\alpha-\beta^2)$$

and the formation between these curves reads as follows:

$$\boldsymbol{z} = \frac{1}{2}(x^2-y-\alpha)\,, \qquad x = \frac{1}{2}\frac{\boldsymbol{w}-\beta}{\boldsymbol{z}-\alpha}\,,$$

$$\boldsymbol{w} = x^3 - yx - 3\alpha x + \beta\,, \qquad y = \frac{1}{4}\frac{(\boldsymbol{w}-\beta)^2}{(\boldsymbol{z}-\alpha)^2} - 2\boldsymbol{z} - \alpha$$

$$= \frac{\beta}{2}\frac{\beta-\boldsymbol{w}}{(\boldsymbol{z}-\alpha)^2} + \frac{1}{4}\frac{9\alpha^2-\gamma}{\boldsymbol{z}-\alpha} - \boldsymbol{z} + \alpha\,.$$

Proof. The variable x, as a function on the curve (77), has two simple poles. If $\mathfrak{u}$ denotes a uniformizer for (77) then we may place these poles at points $\mathfrak{u} = \{0, v\}$ and hence write:

$$\begin{aligned} x &= \zeta(\mathfrak{u}; g_2, g_3) - \zeta(\mathfrak{u}-v; g_2, g_3) - C\,,\\ y &= \wp(\mathfrak{u}-v; g_2, g_3) - \wp(\mathfrak{u}; g_2, g_3)\,, \end{aligned} \tag{78}$$

that is $y = \frac{dx}{d\mathfrak{u}}$. Let us manage parameters g_2, g_3, v, and C to turn (77), (78) into identity in $\mathfrak{u}$. A calculation with use of $\zeta,\wp$-addition theorems yields the constant C and mutual computability of parameters $(\alpha,\beta,\gamma) \leftrightarrows (v, g_2, g_3)$. We obtain that $C = \zeta(v; g_2, g_3)$ and

$$\alpha = \wp(v; g_2, g_3)\,, \quad \beta = \wp'(v; g_2, g_3)\,, \quad \gamma = g_2 - 3\wp^2(v; g_2, g_3)\,,$$

$$v = \wp^{-1}(\alpha; g_2, g_3)\,, \quad g_2 = 3\alpha^2 + \gamma\,, \quad g_3 = \alpha^3 - \gamma\alpha - \beta^2\,. \tag{79}$$

This expressions follow from the fact that pair (α,β) lies on the curve $\beta^2 = 4\alpha^3 - g_2\alpha - g_3$. This also implies the transcendental version

$$y^2 = x^4 - 6\wp(v)x^2 + 4\wp'(v)x + \{g_2 - 3\wp^2(v)\}$$

of algebraic equation (77) and birational isomorphism stated in the proposition is just a $(\wp,\wp')$-version of Eqs. (78) under notation $\boldsymbol{z} = \wp(\mathfrak{u})$ and $\boldsymbol{w} = \wp'(\mathfrak{u})$. ∎

In this connection it should be mentioned Weierstrass' $\wp,\wp'$-formulae in [40, **I**: p. 118–120], [76, **V**], [5], [69, **IV**: p. 66–67], and Biermann's dissertation [10, §1] wherein problem of transition between Weierstrass' cubic and quartic equation (77) was posed for the first time. Complete form to Biermann's birational transformations has been given in *Exs.* 2–3 of [79, 20·6]. Ultimate formulae presented in [10, p. 6] and [79, 20·6] are, however, rather complicated so their simplest form (supplemented with parameters computations) is given by Proposition 22, formulae (78), and Theorem 19. It is know that transformations between two forms of one elliptic curve may contain a free parameter and we can readily introduce it into Proposition 22. To do this it is sufficient to change $\mathfrak{u} \mapsto \mathfrak{u} - \mathfrak{u}_0$ and make use of addition theorems in (re)definitions $\boldsymbol{z} = \wp(\mathfrak{u}-\mathfrak{u}_0)$ and $\boldsymbol{w} = \wp'(\mathfrak{u}-\mathfrak{u}_0)$. Clearly, the $\mathfrak{u}_0$ is more convenient quantity than the 'algebraic parameters' entering into the Biermann–Whittaker–Watson formulae.

The immediate (and known [40, 5]) consequence of the technique above is an application to cubic and quartic equations. With use of Theorem 19 we can present their roots in a completely closed and analytic form. Solutions to cubic $4x^3 - ax - b = 0$ are obvious; these are the Weierstrassian points $x_\kappa = \big\{e(\tau), e'(\tau), e''(\tau)\big\}$ under $\tau = \tau(a,b)$ as above.

Corollary 23. *Closed and radical-free formula for roots x_κ of the quartic equation*

$$x^4-6\alpha x^2+4\beta x+\gamma=0$$

reads as follows

$$x_\kappa=2\zeta\Big(\frac{1}{2}v+\omega_\kappa\Big|\tau\Big)-\zeta(v+2\omega_\kappa|\tau),\qquad \omega_\kappa=\{0,1,\tau,\tau+1\}.$$

Here,

$$v=\wp^{-1}\big(\alpha;3\alpha^2+\gamma,\alpha^3-\gamma\alpha-\beta^2\big),\quad \tau=\mathrm{i}\,\frac{P^{\,0}_{-1/6}\big(-\sqrt{\mathfrak{g}}\big)}{P^{\,0}_{-1/6}\big(\sqrt{\mathfrak{g}}\big)},\quad \mathfrak{g}=27\,\frac{(\alpha^3-\gamma\alpha-\beta^2)^2}{(3\alpha^2+\gamma)^3}$$

and $\wp^{-1}$ is understood to be an elliptic Weierstrassian integral.

To summarize briefly, we may conclude that both transformation between elliptic curves and modular inversion do not require any auxiliary constructions; in each case the ultimate answer is given by an explicit analytic formula independently of Weierstrassian, Legendrian, or general representation (65). Hence, solution to the problem (62) may be thought of as completely solved and we return to integration of the basic equations (50)–(52).

9. Noncanonical θ-functions

9.1. **Exponential quadratic extension of θ-functions.** The system of equations (32) ($\Leftrightarrow$(50)) has fifth order, whereas Weierstrassian base of functions $(\sigma,\zeta,\wp)$ does the order three. In the canonical case $\boldsymbol{A}=\boldsymbol{B}=1$ one can easily derive the differential equation satisfied by any of canonical Jacobi's θ-functions:

$$F_z^2=-4\Big\{F+\frac{\pi^2}{3}(\vartheta_3^4+\vartheta_4^4)\Big\}\Big\{F+\frac{\pi^2}{3}(\vartheta_2^4-\vartheta_4^4)\Big\}\Big\{F-\frac{\pi^2}{3}(\vartheta_2^4+\vartheta_3^4)\Big\},\tag{80}$$

where $F:=\ln_{zz}\theta_k(z|\tau)+4\eta$ and $\vartheta=\vartheta(\tau)$, $\eta=\eta(\tau)$.

In the non-canonical case $\boldsymbol{A}\neq1\neq\boldsymbol{B}$ this equation must have an analog in form of some differential equation of 5th order. It is not difficult to obtain from (50) that each θ-solution of Eqs. (50) satisfies one equation

$$\begin{aligned}&F^2F_{zzz}-2FF_zF_{zz}+F_z^3+(F^4)_z=0,\\ &F:=(\ln\theta)_{zz}+4\Big\{\eta+\frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\Big\}.\end{aligned}\tag{81}$$

This important equation, as equation of 3rd order for function F, is the generalization of a 2nd order differential consequence of the canonical Weierstrassian equation $F_z^2=4F^3-g_2F-g_3$, that is equation $F_{zzz}=12FF_z$. The latter *is not a reduction* of equation (81), although equation (81) is also solved by $\wp$-function:

$$F=\wp(\omega|\omega,\omega')-\wp(z+c|\omega,\omega').$$

with free constants (ω,ω',c). The distinction between them lies in the fact that as long as we do not require the differential closure of θ's, one Weierstrassian equation (80) is sufficient to use. In both of these cases periods $(2\omega,2\omega')$ and modulus τ appear as integration constants. Equation (81) is easily integrated if we rewrite it in the form

$$\bigg(\frac{1}{F_z}\Big(\frac{F_z^2}{F}\Big)_z\bigg)_z+8F_z=0.\tag{82}$$

We thus derive the complete integral of equations (50), whatever parameters ϑ, η. Let

$$M := \varkappa^2\Big\{\eta(\boldsymbol{\tau}) + \frac{\pi^2}{12}\big[\vartheta_3^4(\boldsymbol{\tau}) + \vartheta_4^4(\boldsymbol{\tau})\big]\Big\} - \Big\{\eta + \frac{\pi^2}{12}\big[\vartheta_3^4 + \vartheta_4^4\big]\Big\}.$$

Then some routine computations yield the following result.

Theorem 24. *Differential equations* (32) *and* (50) *have the following general solution*:

$$\begin{aligned}
\pm\theta_1 &= \frac{\vartheta_2\vartheta_3\vartheta_4}{2\boldsymbol{\eta}^3(\boldsymbol{\tau})}\cdot C\theta_1(\varkappa z + B|\boldsymbol{\tau})\,\mathrm{e}^{2M(z+A)^2},\\
\pm\theta_2 &= \frac{\varkappa\vartheta_2}{\vartheta_2(\boldsymbol{\tau})}\cdot C\theta_2(\varkappa z + B|\boldsymbol{\tau})\,\mathrm{e}^{2M(z+A)^2},\\
\pm\theta_3 &= \frac{\varkappa\vartheta_3}{\vartheta_3(\boldsymbol{\tau})}\cdot C\theta_3(\varkappa z + B|\boldsymbol{\tau})\,\mathrm{e}^{2M(z+A)^2},\\
\pm\theta_4 &= \frac{\varkappa\vartheta_4}{\vartheta_4(\boldsymbol{\tau})}\cdot C\theta_4(\varkappa z + B|\boldsymbol{\tau})\,\mathrm{e}^{2M(z+A)^2},\\
\pm\theta_1' &= \frac{\vartheta_2\vartheta_3\vartheta_4}{2\boldsymbol{\eta}^3(\boldsymbol{\tau})}\cdot C\big\{\varkappa\theta_1'(\varkappa z + B|\boldsymbol{\tau}) + 4M(z+A)\theta_1(\varkappa z + B|\boldsymbol{\tau})\big\}\,\mathrm{e}^{2M(z+A)^2},
\end{aligned}\tag{83}$$

where $\{A, B, C, \varkappa, \boldsymbol{\tau}\}$ *is a complete set of integration constants and signs* $\pm$ *may be freely changed for arbitrary pair* (θ_j, θ_k).

Solution (83) shows that its dependence on $\varkappa$ and $\boldsymbol{\tau}$ is rather nontrivial in distinction to dependence on constants B, C and linear exponent e^{Az}; these are easily 'guessable' in solution (54). An additional point to emphasize is that dependence of both equation (81) and its solution (83) on parameters (ϑ, η) is represented, omitting the trivial multiplicative constant C, through the one essential parameter

$$\frac{1}{4}\Lambda = \eta + \frac{\pi^2}{12}\big(\vartheta_3^4 + \vartheta_4^4\big). \tag{84}$$

Using (83), we can, after some algebra, rewrite integrals (57) in the '$(\varkappa, \boldsymbol{\tau})$-representation'.

Corollary 25 (Generalization of Jacobi's identities)**.** *Noncanonical* ϑ, θ*-functions satisfy the following identities*:

$$\vartheta_2^2\theta_4^2 - \vartheta_4^2\theta_2^2 = \varkappa^2\frac{\vartheta_3^4(\boldsymbol{\tau})}{\vartheta_3^4}\cdot\vartheta_3^2\theta_1^2, \qquad \vartheta_2^2\theta_3^2 - \vartheta_3^2\theta_2^2 = \varkappa^2\frac{\vartheta_4^4(\boldsymbol{\tau})}{\vartheta_4^4}\cdot\vartheta_4^2\theta_1^2. \tag{85}$$

The canonical Weierstrass–Jacobi case is defined by the restriction $\boldsymbol{A} = \boldsymbol{B} = 1$ and, therefore, is equivalent to conditions on constants $(\boldsymbol{\tau}, \varkappa)$: $\vartheta(\boldsymbol{\tau}) = \vartheta$, $\eta(\boldsymbol{\tau}) = \eta$, and $\varkappa = \pm 1$.

It should be particularly emphasized the following. Notwithstanding the fact that noncanonical case is realized through the elementary function—the quadratic exponent e^{2Mz^2}—it highly non-elementary depends on constants $(\varkappa, \boldsymbol{\tau})$ and generates a *transcendental extension* since canonical σ- and θ-functions are defined up to a linear exponent by formulae (54). Dependence of the extension on parameters ϑ and η is also nontrivial. Quasi-periodicity properties of the θ_1', θ-extensions, i. e., analogs of formulae (4), are readily established from Eqs. (83) and Theorem 10; we do not display them here.

Remark 9. A brief mention of the quadratic exponential multiplier in front of Jacobian function $\Theta(u)$ can be found in [19, p. 156, 189]. Such a function was also considered

by Jacobi himself; the pages 307–318 of his [48, **I**] are devoted to study of the object $\chi(u) = \mathrm{e}^{ruu}\Omega(u)$, where $\int_0^u E(u)\,du = \log\Omega(u)$ under the standard Jacobi's notation for $E(u)$. The first appearance of the quadratic exponent can be found even in *Fundamenta Nova* [48, **I**: p. 226]. In connection with certain differential identities and heat equation for θ-functions this exponent appears also in [59].

Before going further we pause to comment the character of integrability of Eqs. (32).

9.2. **Algebraic integrability of ODEs for θ-functions.** Equation (82) discloses an interesting feature of canonical and non-canonical θ-series. Let us define the term algebraic integrability as the property of differential equation to have solutions in terms of finitely many integrals of algebraic functions.

Theorem 26. *Differential equations* (32) *are algebraically integrable upon adjoining an inversion of integrals.*

Proof. By virtue of (82) we may write

$$\int^{F}\frac{dx}{\sqrt{x(x-\boldsymbol{a})(x-\boldsymbol{b})}} = 2\mathrm{i}\,z+\boldsymbol{c} \qquad \Rightarrow \qquad F = \Xi(z;\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}), \tag{86}$$

where $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ are some integration constants. Integration is thus completed if we introduce the inversion operation Ξ:

$$\theta = \exp\int^{z}\Big\{\int^{x}\Xi(y;\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\,dy\Big\}\,dx\cdot \mathrm{e}^{-\frac{1}{2}\Lambda z^2+\boldsymbol{d}z+\boldsymbol{e}}, \tag{87}$$

where $\boldsymbol{d},\boldsymbol{e}$ are new constants. The inversion function Ξ here is of course not understood to be a ratio of θ-series. Integration to the θ_1' is obvious. ∎

The two-fold integration of inversion operation in (87) can now be reduced to a 1-fold integration of algebraic function and this leads to a meromorphic integral rather than the holomorphic one as in (86). By this means we obtain the following nonstandard way of introducing the theta-function.

Corollary 27. *The canonical θ-series along with its non-canonical extension can be defined through a meromorphic elliptic integral.*

To prove this it will suffice to make the following change in formula (87):

$$\int^{z}\Xi(y;\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\,dy = \int^{\Xi(z;\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})}\frac{z\,dz}{\sqrt{z(z-\boldsymbol{a})(z-\boldsymbol{b})}}.$$

To all appearances, Tikhomandritskiĭ [71] was the first to point out a way of definition of θ (different from described above) through a meromorphic integral but his old note [71] went unnoticed in the literature.

Theorem 28. *Dynamical systems defining non-canonical extensions of θ-functions* (32)*–*(33) *and ϑ-constants* (56) *are Hamiltonian. They admit the gradient flow forms $\dot{X} = \Omega\nabla H(X)$ with Poisson brackets $\Omega = \Omega(X)$ which may not be the constant ones.*

This theorem enhances results on algebraic integrability (Theorem 26) but its proof and consequences will be given elsewhere because the systems under question require the even-dimensional extensions; these are non-obvious in advance. Here, we give just an example, that, on the one hand, generalizes Hamiltonicity of the canonical version of Eqs. (56) found in [13, Theorem 13] and, on the other hand, is a rational subcase of a pencil of the brackets written out in the same work. Let U, V, and W be defined as three vector field components for system (56) as follows:

$$U(\vartheta,\eta) := \frac{\mathrm{i}}{\pi}\Big\{\eta + \frac{\pi^2}{12}\big(\vartheta_3^4 + \vartheta_4^4 - 3\boldsymbol{B}^4\vartheta_4^4\big)\Big\}\vartheta_3,$$

$$V(\vartheta,\eta) := \frac{\mathrm{i}}{\pi}\Big\{\eta + \frac{\pi^2}{12}\big(\vartheta_3^4 + \vartheta_4^4 - 3\boldsymbol{A}^4\vartheta_3^4\big)\Big\}\vartheta_4,$$

$$W(\vartheta,\eta) := \frac{\mathrm{i}}{\pi}2\eta^2 - \frac{\pi^3}{72}\mathrm{i}\Big\{\vartheta_3^8 + \big(9\boldsymbol{A}^4\boldsymbol{B}^4 - 6\boldsymbol{A}^4 - 6\boldsymbol{B}^4 + 2\big)\vartheta_3^4\vartheta_4^4 + \vartheta_4^8\Big\}$$

and let $\mathfrak{A}$-integral (59) be taken as a Hamiltonian. Then one can show and verify by a straightforward computation the following result.

Theorem 29. *The system* (56) *is Hamiltonian*:

$$\frac{d}{d\tau}\begin{pmatrix}\vartheta_2\\ \vartheta_3\\ \vartheta_4\\ \eta\end{pmatrix} = \frac{\vartheta_2}{4H}\begin{pmatrix}0 & U & V & W\\ -U & 0 & 0 & 0\\ -V & 0 & 0 & 0\\ -W & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}H_{\vartheta_2}\\ H_{\vartheta_3}\\ H_{\vartheta_4}\\ H_\eta\end{pmatrix}.$$

under

$$H(\vartheta_2,\vartheta_3,\vartheta_4,\eta) = \boldsymbol{A}^4\frac{\vartheta_3^4}{\vartheta_2^4} - \boldsymbol{B}^4\frac{\vartheta_4^4}{\vartheta_2^4}.$$

Corresponding Poisson bracket is degenerated $(\det\Omega(\vartheta,\eta)\equiv 0)$ *but single-valued.*

In [14] we also exhibited the fact that algebraic integrability of θ-functions may be treated as a Liouvillian extension of certain differential fields. It therefore has an intimate connection with the Picard–Vessiot solvability of spectral problems defined by linear ODEs.

9.3. **Renormalization of θ-functions.** Solution (83) suggests us to make the following renormalization $\theta \mapsto \boldsymbol{\theta}$:

$$\boldsymbol{\theta}_1 = \theta_1, \quad \boldsymbol{\theta}_2 = \pi\vartheta_3\vartheta_4\cdot\theta_2, \quad \boldsymbol{\theta}_3 = \pi\vartheta_2\vartheta_4\cdot\theta_3, \quad \boldsymbol{\theta}_4 = \pi\vartheta_2\vartheta_3\cdot\theta_4.$$

Then equations (32) and (81) will contain the single parameter (84). We get

$$\begin{gathered}\frac{\partial\boldsymbol{\theta}_1}{\partial z} = \boldsymbol{\theta}_1', \qquad \frac{\partial\boldsymbol{\theta}_1'}{\partial z} = \frac{\boldsymbol{\theta}_1'^2}{\boldsymbol{\theta}_1} - \frac{\boldsymbol{\theta}_2^2}{\boldsymbol{\theta}_1} - \Lambda\cdot\boldsymbol{\theta}_1.\\ \frac{\partial\boldsymbol{\theta}_2}{\partial z} = \frac{\boldsymbol{\theta}_1'}{\boldsymbol{\theta}_1}\boldsymbol{\theta}_2 - \frac{\boldsymbol{\theta}_3\boldsymbol{\theta}_4}{\boldsymbol{\theta}_1}, \qquad \frac{\partial\boldsymbol{\theta}_3}{\partial z} = \frac{\boldsymbol{\theta}_1'}{\boldsymbol{\theta}_1}\boldsymbol{\theta}_3 - \frac{\boldsymbol{\theta}_2\boldsymbol{\theta}_4}{\boldsymbol{\theta}_1}, \qquad \frac{\partial\boldsymbol{\theta}_4}{\partial z} = \frac{\boldsymbol{\theta}_1'}{\boldsymbol{\theta}_1}\boldsymbol{\theta}_4 - \frac{\boldsymbol{\theta}_2\boldsymbol{\theta}_3}{\boldsymbol{\theta}_1}.\end{gathered} \tag{88}$$

It immediately follows that the τ-dependence, that is τ-differentiating the θ-functions (33), is also simplified. Putting for simplicity $\tau = 4\pi\mathrm{i}t$, we obtain the following system of

equations:

$$
\begin{aligned}
&\frac{\partial\boldsymbol{\theta}_1}{\partial t}=\frac{\boldsymbol{\theta}_1'^2}{\boldsymbol{\theta}_1}-\frac{\boldsymbol{\theta}_2^2}{\boldsymbol{\theta}_1}-\Lambda\cdot\boldsymbol{\theta}_1, \qquad \frac{\partial\boldsymbol{\theta}_1'}{\partial t}=\frac{\boldsymbol{\theta}_1'^3}{\boldsymbol{\theta}_1^2}-3\big(\boldsymbol{\theta}_2^2+\Lambda\cdot\boldsymbol{\theta}_1^2\big)\frac{\boldsymbol{\theta}_1'}{\boldsymbol{\theta}_1^2}+2\frac{\boldsymbol{\theta}_2\boldsymbol{\theta}_3\boldsymbol{\theta}_4}{\boldsymbol{\theta}_1^2},\\
&\frac{\partial\boldsymbol{\theta}_2}{\partial t}=\frac{\boldsymbol{\theta}_1'^2}{\boldsymbol{\theta}_1^2}\boldsymbol{\theta}_2-2\boldsymbol{\theta}_1'\frac{\boldsymbol{\theta}_3\boldsymbol{\theta}_4}{\boldsymbol{\theta}_1^2}-\big(\boldsymbol{\theta}_2^2-\boldsymbol{\theta}_3^2-\boldsymbol{\theta}_4^2\big)\frac{\boldsymbol{\theta}_2}{\boldsymbol{\theta}_1^2}-\big\{\Lambda-\ln_t(\vartheta_3\vartheta_4)\big\}\cdot\boldsymbol{\theta}_2,\\
&\frac{\partial\boldsymbol{\theta}_3}{\partial t}=\frac{\boldsymbol{\theta}_1'^2}{\boldsymbol{\theta}_1^2}\boldsymbol{\theta}_3-2\boldsymbol{\theta}_1'\frac{\boldsymbol{\theta}_2\boldsymbol{\theta}_4}{\boldsymbol{\theta}_1^2}+\boldsymbol{\theta}_4^2\frac{\boldsymbol{\theta}_3}{\boldsymbol{\theta}_1^2}-\big\{\Lambda-\ln_t(\vartheta_2\vartheta_4)\big\}\cdot\boldsymbol{\theta}_3,\\
&\frac{\partial\boldsymbol{\theta}_4}{\partial t}=\frac{\boldsymbol{\theta}_1'^2}{\boldsymbol{\theta}_1^2}\boldsymbol{\theta}_4-2\boldsymbol{\theta}_1'\frac{\boldsymbol{\theta}_2\boldsymbol{\theta}_3}{\boldsymbol{\theta}_1^2}+\boldsymbol{\theta}_3^2\frac{\boldsymbol{\theta}_4}{\boldsymbol{\theta}_1^2}-\big\{\Lambda-\ln_t(\vartheta_2\vartheta_3)\big\}\cdot\boldsymbol{\theta}_4.
\end{aligned}
\tag{89}
$$

In the language of normalized equations (88)–(89) the mechanism of integrability for θ-functions (and analysis at all) becomes very simple. Another point that should be emphasized here is an asymmetry of equations. It manifests the fact that we may not use Jacobi's polynomial θ-identities (58) in advance since that identities are just particular integrals of Eqs. (88) and (89); the latter are constructed from the heat equation $\boldsymbol{\theta}_t=\boldsymbol{\theta}_{zz}$ by definition. Furthermore, equations (89) contain not variables ϑ but their logarithmic derivatives. Because of this, compatibility conditions $\boldsymbol{\theta}_{tz}=\boldsymbol{\theta}_{zt}$ for these equations will be 1) *algebraic* relations between functions $\boldsymbol{\theta}$ over field of coefficients Λ, $\ln_t\vartheta$ and, on the other hand, 2) the only differential relation containing $\Lambda_t=:\dot\Lambda$. A computation yields

$$
\begin{gathered}
\frac{\dot\vartheta_2}{\vartheta_2}+\Lambda=0, \qquad \frac{\dot\vartheta_3}{\vartheta_3}+\Lambda=\frac{\boldsymbol{\theta}_3^2-\boldsymbol{\theta}_2^2}{\boldsymbol{\theta}_1^2}, \qquad \frac{\dot\vartheta_4}{\vartheta_4}+\Lambda=\frac{\boldsymbol{\theta}_4^2-\boldsymbol{\theta}_2^2}{\boldsymbol{\theta}_1^2},\\
\dot\Lambda-2\left(\frac{\dot\vartheta_3}{\vartheta_3}+\frac{\dot\vartheta_4}{\vartheta_4}\right)\Lambda-2\frac{\dot\vartheta_3}{\vartheta_3}\frac{\dot\vartheta_4}{\vartheta_4}=0
\end{gathered}
\tag{90}
$$

and we see that parameter $\ln_t\vartheta_2$ is not an independent one but enters into the theory fictitiously, through Λ. The first of Eqs. (90) is in effect the first equation in (56). Right hand sides of 2nd and 3rd of Eqs. (90) are functions of z but their left hand sides are of t. This leads again to algebraic integrals (57):

$$
\boldsymbol{\theta}_3^2-\boldsymbol{\theta}_2^2=\boldsymbol{B}^4\vartheta_4^4\cdot\boldsymbol{\theta}_1^2, \qquad \boldsymbol{\theta}_4^2-\boldsymbol{\theta}_2^2=\boldsymbol{A}^4\vartheta_3^4\cdot\boldsymbol{\theta}_1^2
$$

and 2nd and 3rd equations in (56). The fourth equation is obvious. An important point here is the fact that the primarily object of the theory—compatibility condition—manifests itself not as the symmetrical equations (36) but as non-symmetrical ones (56). Furthermore, because systems (88)–(89) are nonlinear ones, we should use an additional differentiation in (90) to eliminate $\boldsymbol{\theta}$ completely. For symmetry we take the three quantities

$$
2\left(\frac{\dot\vartheta_2}{\vartheta_2},\,\frac{\dot\vartheta_3}{\vartheta_3},\,\frac{\dot\vartheta_4}{\vartheta_4}\right)=:(X,Y,Z)
$$

and obtain at once the following equations:

$$
\dot X=(Y+Z)X-YZ, \qquad \dot Y=(X+Z)Y-XZ, \qquad \dot Z=(X+Y)Z-XY. \tag{91}
$$

This is the famous Darboux–Halphen system [40, **I**: p. 331] and one of its consequence is equivalent to Eq. (39). The scale change $X\mapsto\frac12X$ turns it into equation

$$
\big(\dot X-X^2\big)\dddot X=\ddot X\big(\ddot X-4X^3\big)-2\dot X^2\big(\dot X-3X^2\big). \tag{92}
$$

Applications of system (91) are very well known. See, e.g., works by Ablowitz et all [3], [25, p. 577], Takhtajan [68], and Conte [25, p. 143, 147]. As a vacuum cosmological model these equations come from a particular case of the Bianchi-IX model [21].

Thus, renormalization of θ-functions trivializes the integration scheme of defining ODEs and clarifies interrelations between differential properties of ϑ, θ-functions, heat equation, Darboux–Halphen system, and their consequences. All the equations are integrated in terms of canonical and non-canonical theta-series.

Remark 10. We can continue renormalization $\boldsymbol{\theta} \mapsto \tilde{\boldsymbol{\theta}}$ by putting $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} \exp\big(\frac12 \Lambda z^2\big)$. Then parameter Λ disappears in Eqs. (88) as if we put $\Lambda = 0$ there; then integrability conditions become the simple algebraic relations

$$Y = X + 2\pi^2\varkappa^2\vartheta_4^4(\boldsymbol{\tau}), \qquad Z = X + 2\pi^2\varkappa^2\vartheta_3^4(\boldsymbol{\tau}),$$

and the only differential equation of the Riccati type for function $X(t)$ with variable coefficients $\varkappa = \varkappa(t)$, $\boldsymbol{\tau} = \boldsymbol{\tau}(t)$:

$$\dot X = X^2 + 2\pi^2\varkappa^2\big\{\vartheta_3^4(\boldsymbol{\tau}) + \vartheta_4^4(\boldsymbol{\tau})\big\}\cdot X - 4\pi^4\varkappa^4\vartheta_3^4(\boldsymbol{\tau})\vartheta_4^4(\boldsymbol{\tau}).$$

This equation is also integrable since it is a consequence of equations (90)–(92).

9.4. **General integrals.** Insomuch as equations (91), (39), (92) serve both canonical and non-canonical case, the general integral of equations (56) is a variation of solutions (55). Let us denote the four integration constants for the system (56) as (a, b, c, d). Let us next put $\mathrm{T} := \frac{a\tau+b}{c\tau+\delta}$, where, as usual, $a\delta - bc = 1$, so that δ is not a free parameter.

Theorem 30. *General solution to non-canonical dynamical system (56) is as follows*

$$\begin{gathered}\vartheta_2 = d\frac{\vartheta_2(\mathrm{T})}{\sqrt{c\tau+\delta}}, \qquad \vartheta_3 = \frac{1}{\boldsymbol{A}}\frac{\vartheta_3(\mathrm{T})}{\sqrt{c\tau+\delta}}, \qquad \vartheta_4 = \frac{1}{\boldsymbol{B}}\frac{\vartheta_4(\mathrm{T})}{\sqrt{c\tau+\delta}}, \\ \eta = \frac{1}{(c\tau+\delta)^2}\Big\{\eta(\mathrm{T}) + \frac{\pi^2}{12}\Big[(1-\boldsymbol{A}^{-4})\vartheta_3^4(\mathrm{T}) + (1-\boldsymbol{B}^{-4})\vartheta_4^4(\mathrm{T})\Big]\Big\} + \frac12\frac{\pi i c}{c\tau+\delta},\end{gathered} \tag{93}$$

where $\vartheta_k(\mathrm{T})$ and $\eta(\mathrm{T})$ are understood to be the canonical ϑ, η-series (1) and (8).

Proof. The straightforward verification. ∎

As a corollary we found that the principal parameter of the theory acquires the form

$$\Lambda = \frac{4}{(c\tau+\delta)^2}\Big\{\eta(\mathrm{T}) + \frac{\pi^2}{12}\Big[\vartheta_3^4(\mathrm{T}) + \vartheta_4^4(\mathrm{T})\Big]\Big\} + \frac{2\pi i c}{c\tau+\delta};$$

it contains integration constants but does not do integrals $\boldsymbol{A}$, $\boldsymbol{B}$. Therefore it does not depend on whether canonical or non-canonical case is taken. It also satisfies the 3rd order equation (39) wherein we should put $X = \frac{i}{4\pi}\Lambda$.

In addition to Remark 7 we note that in spite of seemingly simplicity, the symmetrical system (36) is not amenable to integration. Among other things, it is not a compatibility condition for equations (32) and (33) and thus may not be used as an alternative to the correct and integrable system (56). The algebraic integral for the system (36), i.e., (60), is *nonlinear* in variables ϑ^4.

In regard to Remark 4, we note that Jacobi's system (37) is integrated in its full generality along with the system (56). Computations show that Jacobi's $a(h)$ is

$$a = 2I\,\frac{\vartheta_2^4}{\vartheta_3^4}\Big(\frac{\alpha h+\beta}{\gamma h+\delta}\Big)+I \qquad (\alpha\delta-\beta\gamma=1)$$

and remaining functions $b(h)$, $A(h)$, and $B(h)$ are easily computed from equations (37) by differentiating followed by trivial simplification. We thus obtain the complete set of integration constants for Eqs. (37). All this material is discussed at greater length in [13].

Let us assume now that quantities η, ϑ in (32) are functions of τ according to Eqs. (56) and integration constants $\{A, B, C, \varkappa, \boldsymbol{\tau}\}$ are unknown functions of τ. Substituting formulae (83) into (33), we get a system of ODEs for these functions. The calculations can be reduced in advance since we have already had two integrals (57) ($\Leftrightarrow$ (85)):

$$\varkappa\frac{\vartheta_3^2(\boldsymbol{\tau})}{\vartheta_3^2}=\boldsymbol{A}^2, \qquad \varkappa\frac{\vartheta_4^2(\boldsymbol{\tau})}{\vartheta_4^2}=\boldsymbol{B}^2. \tag{94}$$

These relations define a point transformation between algebraic form of integrals $(\boldsymbol{A},\boldsymbol{B})$ and their transcendental counterpart, i.e., the pair $(\varkappa,\boldsymbol{\tau})$; the quantities ϑ_3 and ϑ_4 are parameters here. Hence we can obtain the sought-for equations/solutions by a simpler way. Let the dot above a symbol denote the derivative with respect to τ. Then we derive:

$$\dot{\boldsymbol{\tau}}=\varkappa^2, \quad \pi\mathrm{i}\frac{\dot\varkappa}{\varkappa}=2M, \quad \dot A=0, \quad \dot B=A\dot\varkappa, \quad \frac{\dot C}{C}=-\frac{\dot\varkappa}{\varkappa}.$$

The first two equations have the following solution (implicit in $\boldsymbol{\tau}$)

$$\frac{\vartheta_3(\boldsymbol{\tau})}{\vartheta_4(\boldsymbol{\tau})}=\boldsymbol{p}\left\{\frac{\vartheta_3(\mathrm{T})}{\vartheta_4(\mathrm{T})}\right\}^{\boldsymbol{q}}, \qquad \varkappa=\frac{\sqrt{\boldsymbol{q}}}{c\tau+\delta}\,\frac{\vartheta_2^2(\mathrm{T})}{\vartheta_2^2(\boldsymbol{\tau})}, \tag{95}$$

where $\boldsymbol{p}$, $\boldsymbol{q}$ are new integration constants. The remaining equations are elementary integrated:

$$A=\boldsymbol{E}, \quad B=\frac{\boldsymbol{E}\sqrt{\boldsymbol{q}}}{c\tau+\delta}\,\frac{\vartheta_2^2(\mathrm{T})}{\vartheta_2^2(\boldsymbol{\tau})}+\boldsymbol{D}, \quad C=\frac{\boldsymbol{C}}{\sqrt{\boldsymbol{q}}}\,\frac{\vartheta_2^2(\boldsymbol{\tau})}{\vartheta_2^2(\mathrm{T})}(c\tau+\delta),$$

where $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{E}$ are further integration constants. By virtue of Eqs. (94) and (93) we must put $\boldsymbol{p}=\boldsymbol{q}=1$. Such a reduction of the number of integration constants is dictated by the fact that integrals (94) are integrals of both z- and τ-equations and the equations themselves are nonlinear. The first equation in (95) turns into a relation between $\boldsymbol{\tau}$ and T. One can show that this relation is controlled by the standard modular group $\boldsymbol{\Gamma}(4)$ [57]:

$$\frac{\vartheta_3(\boldsymbol{\tau})}{\vartheta_4(\boldsymbol{\tau})}=\frac{\vartheta_3(\mathrm{T})}{\vartheta_4(\mathrm{T})} \quad\Rightarrow\quad \boldsymbol{\tau}=\widehat{\boldsymbol{\Gamma}(4)}(\mathrm{T}).$$

We choose the simplest case $\boldsymbol{\tau}(\tau)=\mathrm{T}$ and hence the functions $A(\tau)$, $B(\tau)$, $C(\tau)$, and $\varkappa(\tau)$ are immediately determined. We thus arrive at the ultimate answer.

Theorem 31. *Let integrability conditions of Eqs. (32)–(33) or, equivalently, Eqs. (50)–(51), be given by Eqs. (56) and their solution (93). Then the general and simultaneous integral of equations (32) and (33) is as follows:*

$$\pm\theta_1=\frac{1}{\boldsymbol{AB}}\,\frac{d\boldsymbol{C}}{\sqrt{c\tau+\delta}}\,\theta_1\Big(\frac{z+\boldsymbol{E}}{c\tau+\delta}+\boldsymbol{D}\Big|\frac{a\tau+b}{c\tau+\delta}\Big)\,\mathrm{e}^{\frac{-\pi\mathrm{i}c}{c\tau+\delta}(z+\boldsymbol{E})^2},$$

$$\pm\theta_2 = \frac{d\boldsymbol{C}}{\sqrt{c\tau+\delta}}\,\theta_2\Big(\frac{z+\boldsymbol{E}}{c\tau+\delta}+\boldsymbol{D}\Big|\frac{a\tau+b}{c\tau+\delta}\Big)\,\mathrm{e}^{\frac{-\pi\mathrm{i}c}{c\tau+\delta}(z+\boldsymbol{E})^2},$$

$$\pm\theta_3 = \frac{1}{\boldsymbol{A}}\frac{\boldsymbol{C}}{\sqrt{c\tau+\delta}}\,\theta_3\Big(\frac{z+\boldsymbol{E}}{c\tau+\delta}+\boldsymbol{D}\Big|\frac{a\tau+b}{c\tau+\delta}\Big)\,\mathrm{e}^{\frac{-\pi\mathrm{i}c}{c\tau+\delta}(z+\boldsymbol{E})^2},$$

$$\pm\theta_4 = \frac{1}{\boldsymbol{B}}\frac{\boldsymbol{C}}{\sqrt{c\tau+\delta}}\,\theta_4\Big(\frac{z+\boldsymbol{E}}{c\tau+\delta}+\boldsymbol{D}\Big|\frac{a\tau+b}{c\tau+\delta}\Big)\,\mathrm{e}^{\frac{-\pi\mathrm{i}c}{c\tau+\delta}(z+\boldsymbol{E})^2}.$$

Here, $a\delta-bc=1$ and formula for θ_1' is a z-derivative of first of these formulae.

Proof. Straightforward calculation shows that these expressions do indeed solve the systems (32) and (33) under arbitrary constants $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}\}$. Coefficients η and ϑ's contain parameters $\{a,b,c,d\}$ which are free. ∎

Reduction to the canonical case (54) is brought about by putting $\boldsymbol{A}=\boldsymbol{B}=d=1$ and by choosing the transformation $\left(\begin{smallmatrix}a & b\\ c & \delta\end{smallmatrix}\right)=\left(\begin{smallmatrix}1 & 0\\ 2 & 1\end{smallmatrix}\right)\in\boldsymbol{\Gamma}(2)$ since, the group $\boldsymbol{\Gamma}(2)$ does not permute functions θ_k or ϑ_k. Indeed, functions

$$\frac{1}{\sqrt{c\tau+\delta}}\,\theta_k\Big(\frac{z+\boldsymbol{E}}{c\tau+\delta}+\boldsymbol{D}\Big|\frac{a\tau+b}{c\tau+\delta}\Big)\,\mathrm{e}^{\frac{-\pi\mathrm{i}c}{c\tau+\delta}(z+\boldsymbol{E})^2},$$

under the parameters above, become

$$\theta_k \sim \frac{1}{\sqrt{2\tau+1}}\,\theta_k\Big(\frac{z+\boldsymbol{E}}{2\tau+1}+\boldsymbol{D}\Big|\frac{1\tau+0}{2\tau+1}\Big)\,\mathrm{e}^{\frac{-\pi\mathrm{i}2}{2\tau+1}(z+\boldsymbol{E})^2}=\cdots,$$

and, according to Corollary 14,

$$\cdots=\frac{\text{const}}{\sqrt{2\tau+1}}\cdot\sqrt{2\tau+1}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{2\tau+1}\{z+\boldsymbol{E}+\boldsymbol{D}(2\tau+1)\}^2}\,\theta_k\big(z+\boldsymbol{E}+\boldsymbol{D}(2\tau+1)\big|\tau\big)\,\mathrm{e}^{\frac{-2\pi\mathrm{i}}{2\tau+1}(z+\boldsymbol{E})^2}$$
$$=\text{const}\cdot\theta_k\big(z+2\boldsymbol{D}\tau+\boldsymbol{E}+\boldsymbol{D}\big|\tau\big)\,\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{D}\{2z+2\boldsymbol{D}\tau\}},$$

that is the 'linearly exponential' form (54) under $(A,B)=(2\boldsymbol{D},\boldsymbol{E}+\boldsymbol{D})$.

10. An application. The sixth Painlevé transcendent

In addition to applications of the previous machinery mentioned in [12, 13, 14], in this section, we present briefly one more and very nontrivial application to the famous 6th Painlevé equation [25]

$$\begin{aligned} y_{xx}&=\frac12\Big(\frac1y+\frac1{y-1}+\frac1{y-x}\Big)y_x^2-\Big(\frac1x+\frac1{x-1}+\frac1{y-x}\Big)y_x\\ &\quad+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\Big\{\alpha-\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}-\Big(\delta-\frac12\Big)\frac{x(x-1)}{(y-x)^2}\Big\}.\end{aligned}\tag{96}$$

A deep connection of this equation with elliptic functions was established by Painlevé himself in work [60] wherein he gave a remarkable form to Eq. (96):

$$-\frac{\pi^2}{4}\frac{d^2\boldsymbol{z}}{d\tau^2}=\alpha\,\wp'(\boldsymbol{z}|\tau)+\beta\,\wp'(\boldsymbol{z}-1|\tau)+\gamma\,\wp'(\boldsymbol{z}-\tau|\tau)+\delta\,\wp'(\boldsymbol{z}-1-\tau|\tau),\tag{97}$$

by performing the transcendental change of variables $(y,x)\rightleftarrows(\boldsymbol{z},\tau)$:

$$x=\frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)},\qquad y=\frac13+\frac13\frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)}-\frac{4}{\pi^2}\frac{\wp(\boldsymbol{z}|\tau)}{\vartheta_3^4(\tau)}.\tag{98}$$

In 1995 Hitchin [45] found a beautiful solution to Eq. (96) which hitherto remains the most nontrivial of all those currently known. It corresponds to parameters $\alpha=\beta=\gamma=\delta=\frac{1}{8}$ and reads parametrically as follows [45, p. 74, 78]:

$$\wp(\boldsymbol{z}|\tau)=\wp(A\tau+B|\tau)+\frac{1}{2}\,\frac{\wp'(A\tau+B|\tau)}{\zeta(A\tau+B|\tau)-(A\tau+B)\eta(\tau)+\frac{1}{2}\pi\mathrm{i}A}, \tag{99}$$

where A, B are free constants. It is not difficult to show that in the case of that parameters $\{\alpha,\,\beta,\,\gamma,\,\delta\}$ equation (97) can be written in the following θ-function form

$$-\pi^2\frac{d^2\boldsymbol{z}}{d\tau^2}=4\wp'(2\boldsymbol{z}|\tau)\quad\Leftrightarrow\quad\frac{d^2\boldsymbol{z}}{d\tau^2}=4\pi\boldsymbol{\eta}^9(\tau)\,\frac{\theta_1(2\boldsymbol{z}|\tau)}{\theta_1^4(\boldsymbol{z}|\tau)}.$$

With complete rules of differential θ-computations in hand, we can obtain their analog for Weierstrassian functions and thereby automate and simplify manipulation with all solutions to Eq. (96) expressible in terms of elliptic or θ-functions.

10.1. **Weierstrass' functions and Hitchin's solution.** In order to derive rules for derivatives of Weierstrassian functions with respect to half-periods (ω,ω') one can use the preceding θ-apparatus supplemented with rules (19), (29)–(30), or, alternatively, transformations between derivatives in (g_2,g_3) and (ω,ω') [34, p. 263–265]. The (g_2,g_3)-derivatives were considered by Weierstrass [77, **V**], [40, **I**], [69, **IV**] and, at about the same time, by Frobenius & Stickelberger [37]. Let us denote a sign of the period ratio as $\mathfrak{s}:=\mathfrak{sign}\Big\{\Im\big(\frac{\omega'}{\omega}\big)\Big\}$. Applying now any of the techniques above and carrying out some simplification, we obtain that the sought-for formulae acquire very compact and symmetrical form.

Theorem 32. *Rules for differentiating Weierstrass' $(\sigma,\zeta,\wp,\wp')$-functions are*

$$\left\{\begin{aligned}\mathfrak{s}\,\frac{\partial\sigma}{\partial\omega}&=-\frac{\mathrm{i}}{\pi}\Big\{\omega'\Big(\wp-\zeta^2-\tfrac{1}{12}g_2\,z^2\Big)+2\eta'\,(z\,\zeta-1)\Big\}\,\sigma\\ \mathfrak{s}\,\frac{\partial\sigma}{\partial\omega'}&=\ \ \frac{\mathrm{i}}{\pi}\Big\{\omega\Big(\wp-\zeta^2-\tfrac{1}{12}g_2\,z^2\Big)+2\eta\,(z\,\zeta-1)\Big\}\,\sigma\end{aligned}\right.,$$

$$\left\{\begin{aligned}\mathfrak{s}\,\frac{\partial\zeta}{\partial\omega}&=-\frac{\mathrm{i}}{\pi}\Big\{2\,(\omega'\zeta-z\,\eta')\,\wp+\omega'\Big(\wp'-\tfrac{1}{6}g_2\,z\Big)+2\eta'\zeta\Big\}\\ \mathfrak{s}\,\frac{\partial\zeta}{\partial\omega'}&=\ \ \frac{\mathrm{i}}{\pi}\Big\{2\,(\omega\,\zeta-z\,\eta)\,\wp+\omega\Big(\wp'-\tfrac{1}{6}g_2\,z\Big)+2\eta\,\zeta\Big\}\end{aligned}\right.,$$

$$\left\{\begin{aligned}\mathfrak{s}\,\frac{\partial\wp}{\partial\omega}&=\ \ \frac{\mathrm{i}}{\pi}\Big\{2\,(\omega'\zeta-z\,\eta')\,\wp'+4\,(\omega'\wp-\eta')\,\wp-\tfrac{2}{3}\omega' g_2\Big\}\\ \mathfrak{s}\,\frac{\partial\wp}{\partial\omega'}&=-\frac{\mathrm{i}}{\pi}\Big\{2\,(\omega\,\zeta-z\,\eta)\,\wp'+4\,(\omega\,\wp-\eta)\,\wp-\tfrac{2}{3}\omega\, g_2\Big\}\end{aligned}\right.,$$

$$\left\{\begin{aligned}\mathfrak{s}\,\frac{\partial\wp'}{\partial\omega}&=\ \ \frac{\mathrm{i}}{\pi}\Big\{6\,(\omega'\wp-\eta')\,\wp'+(\omega'\zeta-z\,\eta')(12\,\wp^2-g_2)\Big\}\\ \mathfrak{s}\,\frac{\partial\wp'}{\partial\omega'}&=-\frac{\mathrm{i}}{\pi}\Big\{6\,(\omega\,\wp-\eta)\,\wp'+(\omega\,\zeta-z\,\eta)(12\,\wp^2-g_2)\Big\}\end{aligned}\right..$$

Setting in these equations $\omega = 1$, $\omega' = \tau$, and $\mathfrak{s} = 1$, we arrive at a dynamical system containing parameter z:

$$\left\{\begin{aligned} \frac{\partial\sigma}{\partial\tau} &= \frac{\mathrm{i}}{\pi}\Big\{\wp - \zeta^2 + 2\eta\,(z\zeta - 1) - \frac{1}{12}\,g_2\,z^2\Big\}\,\sigma \\ \frac{\partial\zeta}{\partial\tau} &= \frac{\mathrm{i}}{\pi}\Big\{\wp' + 2(\zeta - z\eta)\,\wp + 2\eta\zeta - \frac{1}{6}\,g_2\,z\Big\} \\ \frac{\partial\wp}{\partial\tau} &= -\frac{\mathrm{i}}{\pi}\Big\{2(\zeta - z\eta)\,\wp' + 4(\wp - \eta)\,\wp - \frac{2}{3}\,g_2\Big\} \\ \frac{\partial\wp'}{\partial\tau} &= -\frac{\mathrm{i}}{\pi}\big\{6(\wp - \eta)\,\wp' + (\zeta - z\eta)(12\,\wp^2 - g_2)\big\} \end{aligned}\right. , \tag{100}$$

where we use the right brace additionally to denote the differential closedness of the functions $(\zeta, \wp, \wp')$. It follows that the triple of functions $\zeta(z|\tau)$, $\wp(z|\tau)$, and $\wp'(z|\tau)$ is differentially closed with respect to both variables z and τ. We have also to close the derivatives of coefficients g_2 and η. This is realized by the Halphen system (35) but third variable g_3 is not present in system (100) or in the system of z-equations

$$\frac{d\zeta}{dz} = -\wp, \qquad \frac{d\wp}{dz} = \wp', \qquad \frac{d\wp'}{dz} = 6\,\wp^2 - \frac{1}{2}\,g_2\,. \tag{101}$$

Therefore we may treat the classical Weierstrassian relation between $\wp$ and $\wp'$

$$g_3(\wp, \wp') = 4\,\wp^3 - g_2\,\wp - \wp'^2$$

as the *algebraic (polynomial) integral* of equations (101) or as the *surface of a constant level* for the dynamical system with variable coefficients, i. e., system (100).

Another corollary of this system is the fact that each of functions ζ, $\wp$, $\wp'$ satisfies the ordinary τ-differential equation of 2nd order with variable coefficients g_2, g_3, η, and function σ does an equation of 3rd order. These equations are too large to display here. The function $\boldsymbol{Z} = \zeta(z|\tau) - z\eta(\tau)$, as an example, solves an equation obtainable by elimination of variable $\wp$ from the two polynomials being consequences of (100):

$$\big\{\pi\mathrm{i}\,\boldsymbol{Z}_\tau + 2(\wp + \eta)\,\boldsymbol{Z}\big\}^2 - 4\,\wp^3 + g_2\,\wp + g_3\,,$$

$$\frac{\pi^2}{8}\,\frac{\boldsymbol{Z}_{\tau\tau}}{\boldsymbol{Z}} + \mathrm{i}\,\frac{\pi}{2}\,(\boldsymbol{Z}^2 + \wp - 2\eta)\,\frac{\boldsymbol{Z}_\tau}{\boldsymbol{Z}} + (\wp + \eta)\,\boldsymbol{Z}^2 - \wp^2 + \eta\,\wp - \eta^2 + \frac{1}{4}\,g_2\,;$$

these polynomials are understood to be equal to zero. They contain no variable z explicitly. We do not consider here the θ, θ'-analogs of these equations.

We observe that Hitchin's solution is a function of the quantities $\zeta(A\tau + B|\tau)$, $\wp(A\tau + B|\tau)$, and $\wp'(A\tau + B|\tau)$. It follows that these three functions define a dynamical system in its own right that follows from equations (100)–(101).

Proposition 33. *The Hitchin case of Painlevé equation (96) is equivalent to the dynamical system*

$$\frac{d\boldsymbol{\zeta}}{d\tau} = \frac{\mathrm{i}}{\pi}\big\{\boldsymbol{\wp}' + 2(\boldsymbol{\wp} + \eta)\,\boldsymbol{\zeta}\big\}, \qquad \frac{d\boldsymbol{\wp}}{d\tau} = -\frac{\mathrm{i}}{\pi}\Big\{2\,\boldsymbol{\zeta}\,\boldsymbol{\wp}' + 4(\boldsymbol{\wp} - \eta)\,\boldsymbol{\wp} - \frac{2}{3}\,g_2\Big\} \tag{102}$$

with variable coefficients $\eta = \eta(\tau)$, $g_2 = g_2(\tau)$, $g_3 = g_3(\tau)$. Here, $\boldsymbol{\wp}' := \sqrt{4\boldsymbol{\wp}^3 - g_2\boldsymbol{\wp} - g_3}$, and equations (102) may be supplemented by their corollary:

$$\frac{d\boldsymbol{\wp}'}{d\tau} \equiv -\frac{\mathrm{i}}{\pi}\big\{6(\boldsymbol{\wp} - \eta)\,\boldsymbol{\wp}' + (12\,\boldsymbol{\wp}^2 - g_2)\,\boldsymbol{\zeta}\big\}\,.$$

General integral of the equations reads

$$\boldsymbol{\zeta} = \zeta(A\tau + B|\tau) - A\eta'(\tau) - B\eta(\tau), \qquad \boldsymbol{\wp} = \wp(A\tau + B|\tau). \tag{103}$$

In other words, we may view the sixth Painlevé transcendent under Hitchin's parameters as a pair of 'τ-equations' (102), i. e., as an integrable case of the τ-representation (97).

One further consequence of the results above is that we can construct the Painlevé–Hitchin integral τ-calculus. Indeed, the first of equations (100) suggests that the quantity $\wp - (\zeta - z\eta)^2$ is integrable with respect to τ and expressible through a logarithm of σ-function. Namely,

$$-\pi\mathrm{i}\frac{d}{d\tau}\ln\sigma = \wp - (\zeta - z\eta)^2 + \Big(\eta^2 - \frac{1}{12}g_2\Big)z^2 - 2\eta. \tag{104}$$

Owing to equations (102) and the fact that Hitchin's solution have a z-argument in form $z = A\tau + B$, we can extend (104) and consider the objects (103), i. e., $\boldsymbol{\wp} - \boldsymbol{\zeta}^2$. Therefore

$$-\pi\mathrm{i}\frac{d}{d\tau}\ln\sigma(A\tau + B|\tau) = \boldsymbol{\wp} - \boldsymbol{\zeta}^2 + f(\tau),$$

where unknown function $f(\tau)$ is independent of $\boldsymbol{\zeta}$ and $\boldsymbol{\wp}$. It is readily determined by a differentiation followed by use of differential connection (12) between $\boldsymbol{\eta}$ and η. Applying an antiderivative to the last equation, we obtain the nice indefinite integral

$$\frac{\mathrm{i}}{\pi}\int^{\tau}(\boldsymbol{\wp} - \boldsymbol{\zeta}^2)\,d\tau = \ln\theta_1\big(\tfrac12 A\tau + \tfrac12 B\big|\tau\big) - \ln\boldsymbol{\eta}(\tau) + \frac14\pi\mathrm{i}A^2\tau;$$

it can be checked by a straightforward differentiation and conversion everything to ϑ, θ-functions. This integral is nothing but the Painlevé τ-analog of Weierstrassian relation $\int\zeta\,dz = \ln\sigma$ between meromorphic ζ-function and entire function σ.

10.2. θ-function forms to Hitchin's solution. In addition to solution (99), Hitchin suggested also its θ-function form. We reproduce it here in original notation of the work [45, p. 33]:

$$\begin{aligned} y(x) = {}& \frac{\vartheta_1'''(0)}{3\pi^2\vartheta_4^4(0)\,\vartheta_1'(0)} + \frac13\left(1 + \frac{\vartheta_3^4(0)}{\vartheta_4^4(0)}\right) \\ &+ \frac{\vartheta_1'''(\nu)\,\vartheta_1(\nu) - 2\vartheta_1''(\nu)\,\vartheta_1'(\nu) + 4\pi\mathrm{i}c_1\,(\vartheta_1''(\nu)\,\vartheta(\nu) - \vartheta_1'^2(\nu))}{2\pi^2\vartheta_4^4(0)\,\vartheta_1(\nu)\,(\vartheta_1'(\nu) + 2\pi\mathrm{i}c_1\,\vartheta_1(\nu))}, \end{aligned}$$

where $\nu = c_1\tau + c_2$. Differential properties of θ-functions or conversion formulae like (29)–(30) say that the availability of the higher θ-derivatives is excessive here and we can simplify this solution. Doing this, we obtain the very simple formula

$$y = \frac{\sqrt{x}}{\theta_1^2}\left\{\frac{\pi\vartheta_2^2\cdot\theta_2\theta_3\theta_4}{\theta_1' + 2\pi A\theta_1} - \theta_2^2\right\},$$

where symbols ϑ, θ_1', θ are understood to be

$$\theta_1' = \theta_1'\Big(A\tfrac{K(\sqrt{x})}{K'(\sqrt{x})} + B\Big|\tfrac{\mathrm{i}K(\sqrt{x})}{K'(\sqrt{x})}\Big), \quad \theta_k = \theta_k\Big(A\tfrac{K(\sqrt{x})}{K'(\sqrt{x})} + B\Big|\tfrac{\mathrm{i}K(\sqrt{x})}{K'(\sqrt{x})}\Big), \quad \vartheta_2 = \vartheta_2\Big(\tfrac{\mathrm{i}K(\sqrt{x})}{K'(\sqrt{x})}\Big).$$

Elliptic integrals K and K' were introduced above. They give an inversion of the first formula in (98), that is an equivalent of Jacobi's formula (67):

$$\tau = \mathrm{i}\,\frac{K(\sqrt{x})}{K'(\sqrt{x})}\,.$$

Further reading of Hitchin' solution is related to the fact that all solutions to the Painlevé equations are meromorphic functions with fixed branch points. For equation (96) these are the three points $x = \{0, 1, \infty\}$ and the general theory of Painlevé equations guaranties availability of what is called the $\boldsymbol{\tau}$-representation [25, p. 165]

$$y \sim x(1-x)\frac{d}{dx}\ln\frac{\boldsymbol{\tau}_1}{\boldsymbol{\tau}_2} \tag{105}$$

Here, the 'bold tau' is a traditional tau-function notation having nothing in common with modulus τ or modulus $\boldsymbol{\tau}$ in sects. 7.2 and 9. Equations (100) have the following consequence

$$\frac{\pi}{2\mathrm{i}}\frac{d}{d\tau}\ln\bigl\{\zeta(z|\tau) - z\eta(\tau)\bigr\} = \wp(z|\tau) + \frac{1}{2}\frac{\wp'(z|\tau)}{\zeta(z|\tau) - z\eta(\tau)} + \eta(\tau)\,.$$

Comparing this property with (99), we observe the total logarithmic derivative

$$\wp(z|\tau) = \frac{\pi}{2\mathrm{i}}\frac{d}{d\tau}\ln\frac{\zeta(A\tau+B|\tau) - A\eta'(\tau) - B\eta(\tau)}{\boldsymbol{\eta}^2(\tau)}\,.$$

Replacing (A, B) with $(2A, 2B)$ and transforming the right-hand side of this equation into the θ-functions, we can rewrite the previous parametric form of the solution as follows:

$$y = \frac{2\mathrm{i}}{\pi}\frac{1}{\vartheta_3^4(\tau)}\frac{d}{d\tau}\ln\frac{\theta_1'(A\tau+B|\tau) + 2\pi\mathrm{i}A\theta_1(A\tau+B|\tau)}{\vartheta_2^2(\tau)\theta_1(A\tau+B|\tau)}\,, \qquad x = \frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)}\,.$$

Conversion of this result into the original x-representation (105) becomes now an exercise because all the differential calculus has been described completely.

Proposition 34. *The general solution to the Hitchin's case of Painlevé equation* (96) *in the tau-function form* (105) *is as follows*:

$$\begin{aligned}
y &= x(1-x)\frac{d}{dx}\ln\frac{\Bigl\{\theta_1'\Bigl(A\frac{K(\sqrt{x})}{K'(\sqrt{x})} + B\Bigl|\frac{\mathrm{i}K(\sqrt{x})}{K'(\sqrt{x})}\Bigr) + 2\pi A\cdot\theta_1\Bigl(A\frac{K(\sqrt{x})}{K'(\sqrt{x})} + B\Bigl|\frac{\mathrm{i}K(\sqrt{x})}{K'(\sqrt{x})}\Bigr)\Bigr\}^2}{(1-x)\,\theta_1^2\Bigl(A\frac{K(\sqrt{x})}{K'(\sqrt{x})} + B\Bigl|\frac{\mathrm{i}K(\sqrt{x})}{K'(\sqrt{x})}\Bigr)K'^2(\sqrt{x})} \\
&= \frac{E'(\sqrt{x})}{K'(\sqrt{x})} + 2x(1-x)\frac{d}{dx}\ln\Bigl\{\frac{\theta_1'}{\theta_1}\Bigl(A\frac{K(\sqrt{x})}{K'(\sqrt{x})} + B\Bigl|\frac{\mathrm{i}K(\sqrt{x})}{K'(\sqrt{x})}\Bigr) + 2\pi A\Bigr\},
\end{aligned}$$

where Legendre's elliptic integrals (K, K', E, E') [32, 79], *as functions of* $\sqrt{x}$, *are differentially closed according to the rules*

$$2\frac{dK}{dx} = \frac{E}{x(1-x)} - \frac{K}{x}, \qquad 2\frac{dK'}{dx} = \frac{E'}{x(x-1)} - \frac{K'}{x-1},$$

$$2\frac{dE}{dx} = \frac{E}{x} - \frac{K}{x}, \qquad 2\frac{dE'}{dx} = \frac{E'}{x-1} - \frac{K'}{x-1}.$$

Verifying this form of solution by a direct substitution in (96) is a good and rather nontrivial exercise. We do not go into further details because comprehensive analysis of this $\boldsymbol{\tau}$-function form, including additional motivation, explanations, and corollaries, have been detailed in [11]. In the same place the complete reference list can be found.

References

[1] Ablowitz, M. J., Chakravarty, S. & Halburd, R. *The Generalized Chazy Equation from the Self-Duality Equations.* Stud. Appl. Math. **103**(1) (1999), 75–88.

[2] Ablowitz, M. J., Chakravarty, S. & Hahn, H. *Integrable systems and modular forms of level* 2. J. Phys. **A**: Math. Gen. **39**(50) (2006), 15341–15353.

[3] Ablowitz, M. J., Chakravarty, S. & Takhtajan, L. A. *A self-dual Yang–Mills hierarchy and its reduction to integrable systems in* 1+1 *and* 2+1 *dimensions.* Comm. Math. Phys. **158** (1993), 289–314.

[4] Abramowitz, M. & Stegan, I. *Handbook of mathematical functions.* National Bureau of Standards: Appl. Math. Series **55** (1964).

[5] Akhiezer, N. I. *Elements of the Theory of Elliptic Functions.* Transl. Math. Monogr. **79**, Amer. Math. Soc., Providence, R. I. (1990).

[6] Apostol, N. M. *Modular functions and Dirichlet Series in Number Theory.* 2nd ed. Springer–Verlag: New York (1990).

[7] Atiyah, M. & Hitchin, N. *The Geometry And Dynamics of Magnetic Monopoles.* Princeton University Press: Princeton (1988).

[8] Baruch, A. *Ueber die Differentialrelationen zwischen den Thetafunktionen eines Arguments.* Inaugural-Dissertation, Halle. Buchdruckerel von W. Fr. Kaestner: Göttingen (1910), 35 pp.

[9] Belokolos, E. D., Bobenko, A. I., Enol'skii, V. Z., Its, A. R. & Matveev, V. B. *Algebro-Geometric Approach to Nonlinear Integrable Equations.* Springer–Verlag (1994).

[10] Biermann, G. G. A. *Probelemata quaedam mechanica functionum ellipticarum ope soluta.* Dissertatio Inauguralis. Typis Caroli Schultze: Universität Berlin (1865), 39 pp.

[11] Brezhnev, Yu. V. *A $\boldsymbol{\tau}$-function solution of the sixth Painlevé transcendent.* Theor. Math. Phys. **161**(3) (2009), 1616–1633.

[12] Brezhnev, Yu. V. *Transcendental trace formulas for finite-gap potentials.* Theor. Math. Phys. **164**(1) (2010), 920–928.

[13] Brezhnev, Yu. V., Lyakhovich, S. L. & Sharapov, A. A. *Dynamical systems defining Jacobi's ϑ-constants.* Journ. Math. Phys. **52**(11) (2011), 112704(1–21).

[14] Brezhnev, Yu. V. *Spectral/quadrature duality: Picard–Vessiot theory and finite-gap potentials.* Contemporary Math. **563** (2012), 1–31. Amer. Math. Soc.

[15] Briot, C. & Bouquet, C. *Theorie des fonctions elliptiques.* 2-nd ed. Gauthier–Villars: Paris (1875).

[16] Bruinier, J. H., van der Geer, G., Harder, G. & Zagier, D. *The* 1-2-3 *of Modular Forms.* Springer: Berlin–Heidelberg (2008).

[17] Buchstaber, V. M, Leykin, D. V. & Pavlov, M. V. *Egorov Hydrodynamic Chains, the Chazy Equation and $SL(2,\mathbb{C})$.* Funct. Anal. Appl. **37** (2003), 251–262.

[18] Cayley, A. *An Elementary Treatise on Elliptic Functions.* 2nd ed. George Bell and Sons: London (1895). Reprint: Dover (1961).

[19] Cayley, A. *The collected mathematical papers.* ***X***. Cambridge University Press: Cambridge (1896).

[20] Chakravarty, S., Ablowitz, M. J & Clarkson, P. A. *Reductions of self-dual Yang–Mills fields and classical systems.* Phys. Rev. Lett. **65**(9) (1990), 1085–1087.

[21] Chakravarty, S. & Ablowitz, M. J. *Integrability, Monodromy, Evolving Deformations, and Self-Dual Bianchi IX Systems.* Phys. Rev. Lett. **76**(6) (1996), 857—860 (1996).

[22] Chandrasekharan, K. *Elliptic Functions.* Springer–Verlag: Berlin–Heidelberg (1985).

[23] Chazy, J. *Sur les équations différentielles dont l'intégrale générale possède une coupure essentielle mobile.* Compt. Rend. Acad. Sci. Paris **150** (1910), 456–458.

[24] Clarkson, P. A. & Olver, P. J. *Symmetry and the Chazy equation.* J. Diff. Equations **124**(1) (1996), 225–246.

[25] Conte, R. (Ed.) *The Painlevé property. One century later.* CRM Series in Mathematical Physics. Springer–Verlag: New York (1999).

[26] Cox, D., Little, J. & O'Shea, D. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra.* (3rd ed.) Springer: New York (2007).

[27] Durège, H. *Theorie der elliptischen Funktionen.* Verlag von B. G. Teubner: Leipzig (1908).

[28] Eilbeck, J. C. & Enol'skii, V. Z. *Bilinear operators and power series for the Weierstrass σ-function.* Journ. Phys. **A**: Math. Gen. **33**(4) (2000), 791–794.

[29] Eisenstein, G. *Mathematische Abhandlungen.* Georgs Olms Verlagsbuchhandlung: Hildesheim (1967).

[30] Enneper, A. *Elliptische Functionen. Theorie und Geschichte.* Verlag von Louis Nebert: Halle (1876).

[31] Erdélyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F. G. *Higher Transcendental Functions **I**. The Hypergeometric Function, Legendre Functions.* McGraw–Hill: New York (1953).

[32] Erdélyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F. G. *Higher Transcendental Functions **III**. Elliptic and automorphic functions.* McGraw–Hill: New York (1955).

[33] Farkas, H. & Kra, I. *Theta Constants, Riemann Surfaces and the Modular Group.* Graduate Studies in Math. **37**, AMS (2001).

[34] Forsyth, A. R. *Theory of functions of a complex variable.* Cambridge University Press: Cambridge (1893).

[35] Fricke, R. *Elliptische Funktionen.* In: *Encyklopädie der mathematischen Wissenschaften. **II**.* 2. Teil. (1913), 181–348. Verlag und Druck von B. G. Teubner: Leipzig (1913).

[36] Fricke, R. *Die elliptischen funktionen und ihre anwendengen. **I–II**.* Druck und Verlag von B. G. Teubner: Leipzig und Berlin (1916), (1922).

[37] Frobenius, F. & Stickelberger, L. *Ueber die Differentiation der elliptischen Functionen nach den Perioden und Invarianten.* Journ. Reine Angew. Math. **XCII** (1882), 311–337.

[38] Fuchs, R. *Sur quelques équations différentielles linéares du second ordre.* Compt. Rend. Acad. Sci. **CXLI**(14) (1905), 555–558.

[39] Gudermann, Ch. *Theorie der Modular-Functionen und der Modular-Integrale.* G. Reiner: Berlin (1844).

[40] Halphen, G.-H. *Traité des Fonctions Elliptiques et de Leurs Applications. **I–III**.* Gauthier–Villars: Paris (1886)–(1891).

[41] Hancock, H. *Lectures on the theory of elliptic functions. **I**.* J. Weyley & Sons: New York, Chapman & Hall: London (1910).

[42] Harnad, J. *Picard–Fuchs Equations, Hauptmodulus and Integrable Systems.* In: *Integrability: The Seilberg–Witten and Witham Equations.* (Edinburgh, 1998). Ch. 8, 137–151 (2000).

[43] Harnad, J. & McKay, J. *Modular solutions to equations of generalized Halphen type.* Proc. Royal Soc. London **A 456** (2000), 261–294.

[44] Hermite, C. *Œuvres de Charles Hermite. **II**.* Gauthier–Villars: Paris (1908).

[45] Hitchin, N. *Twistor spaces, Einstein metrics and isomonodromic deformations.* Journ. Diff. Geom. **42**(1) (1995), 30–112.

[46] Hurwitz, A. *Grundlagen einer independenten Theorie der elliptischen Modulfunctionen und Theorie der Multiplicatorgleichungen erster Stufe.* Math. Ann. **XVIII**(3) (1881), 528–592.

[47] Hurwitz, A. & Courant, R. *Vorlesungen üeber algemeine funktionentheorie und elliptische funktionen.* Springer–Verlag (1964).

[48] Jacobi, C. G. J. *Gesammelte Werke. **I, II**.* Verlag von G. Reimer: Berlin (1882).

[49] Klein, F. & Fricke, R. *Vorlesungen Ëuber die Theorie der elliptishen Modulfunktionen. **I**: Grundlegung der Theorie; Fortbildung und Anwendung der Theorie. **II**.* Teubner: Leipzig (1890–1892).

[50] Koenigsberger, L. *Vorlesungen über die theorie der Elliptischen Functionen. **II**.* Druck und Verlag von B. G. Teubner: Leipzig (1874).

[51] Koenigsberger, L. *Zur Geschichte der Theorie der elliptischen Transzendenten in den Jahren* 1826–29. Druck und Verlag von B. G. Teubner: Leipzig (1879).

[52] Krause, M. *Theorie der Doppeltperiodischen Functionen. **I**.* Verlag von B. G. Teubner: Leipzig (1895).

[53] Krazer, A. *Lehrbuch der Thetafunktionen.* Druck und Verlag von B. G. Teubner: Leipzig (1903).

[54] Lawden, D. F. *Elliptic Functions and Applications.* Applied Mathematical Sciences **80**. Springer–Verlag: New York (1989).

[55] Levin, A. *A Geometric Interpretation of an Infinite Product for the Lemniscate Constant.* Amer. Math. Month. **113** (2006), 510–520.

[56] Maier, R. S. *Nonlinear differential equations satisfied by certain classical modular forms.* Manuscripta Mathematica **134**(1/2) (2011), 1–42.

[57] Mumford, D. *Tata Lectures on Theta. **I–II**.* Progress in Math. **28**, **43**. Birkhäuser: Boston (1983)–(1984).

[58] Neumann, C. *Vorlesungen über Riemann's Theorie der Abeleschen Integrale.* 2nd ed. Teubner: Leipzig (1884); 1st ed. (1865).

[59] Ohyama, Y. *Differential relations of theta functions.* Osaka J. Math. **32**(2) (1995), 431–450.

[60] Painlevé, P. *Sur les équations différentialles du second ordre à points critiques fixes.* Compt. Rend. Acad. Sci. **CXLIII**(26) (1906), 1111–1117.

[61] Rademacher, H. & Grosswald, E. *Dedekind sums.* The Carus Math. Monographs **16** (1972).

[62] Ramanujan, S. *On certain arithmetical functions.* Trans. Cambridge Phil. Soc. **22**(9) (1916), 159–184; *Collected papers.* Cambridge University Press (1927).

[63] Rankin, R. A. *Modular forms and functions.* Cambridge University Press: Cambridge (1977).

[64] Rauch, H. E. & Lebowitz, A. *Elliptic Functions, Theta Functions, and Riemann Surfaces.* The Williams & Wilkins Comp. (1973).

[65] Riemann, B. *Elliptische Functionen.* Herausgegeben von H. Stahl. Druck und Verlag von B. G. Teubner: Leipzig (1899).

[66] Rosati, L. & Nucci, M. C. *A Lie Symmetry Connection between Jacobi's Modular Differential Equation ans Schwarzian Differential Equation.* J. Nonlin. Math. Phys. **12**(2) (2005), 144–161.

[67] Сомовъ, I. *Основанія теоріи эллиптическихъ функцій.* Типографія Императорской Академіи Наукъ: С.-Петербургъ (1850). Somov, I. *Foundations for the theory of elliptic functions.* The Imperial Academy of Sciences: St. Petersburg (1850) (in old Russian).

[68] Takhtajan, L. A. *A simple example of modular forms as tau-functions for integrable equations.* Theor. Math. Phys. **93**(2) (1992), 330–341.

[69] Tannery, J. & Molk, J. *Éléments de la théorie des fonctions elliptiques.* ***I–IV***. Gauthier–Villars: Paris (1893–1902).

[70] Thomae, J. *Sammlung von Formeln und Sätzen aus dem Gebiete der elliptischen Funktionen nebst Anwendungen.* Verlag von B. G. Teubner: Berlin (1905).

[71] Тихомандрицкій, M. A. *Замѣтка о введеніи Θ-функцій въ теорію эллиптическихъ функцій.* Сообщенія и протоколы заседаній математическаго общества при Императорскомъ Харьковскомъ Университетѣ (1883), Сер. I, 47–67. Tikhomandritskiĭ, M. A. *A note on introduction of* Θ-*functions to the theory of elliptic functions.* Communications and proceedings of Math. Soc. Khar'kov Imperial Univ. (1883), Ser. I, 47–67 (in Russian). Brief version of this note: Tichomandritzky, M. *Ueber das Umkehrproblem der elliptischen Integrale.* Math. Ann. **XXII**(3) (1883), 450–454.

[72] Тихомандрицкій, M. A. *Теорія эллиптическихъ интеграловъ и эллиптическихъ функцій.* Типографія Зильберберга: Харьковъ (1895). Tikhomandritzky, M. A. *Théorie des intégrales et des fonctions elliptiques.* Khar'kov (1895) (in Russian).

[73] Tod, K. P. *Self-dual Einstein metrics from the Painlevé VI equation.* Phys. Lett. **A 190**(3–4) (1994), 221–224.

[74] Todd, J. *The Lemniscate Constants.* Commun. Assoc. Comp. Machinery **18**(1) (1975), 14–19; corrigendum, ibid. **18**(8), 462.

[75] Weber, H. *Lehrbuch der Algebra.* ***III***. *Elliptische Funktionen und algebraische Zahlen.* F. Vieweg & Sohn: Braunschweig (1908).

[76] Weierstrass, K. *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen.* Bearbeitet und herausgegeben von H. A. Schwarz. W. Fr. Kaestner: Göttingen (1885).

[77] Weierstrass, K. *Mathematische Werke.* ***II, V***. Mayer & Müller: Berlin (1895), (1904).

[78] Weil, A. *Elliptic functions according to Eisenstein and Kronecker.* Springer–Verlag: Berlin–Heidelberg (1976).

[79] Whittaker, E. T. & Watson, G. N. *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions, with an Account of the Principal Transcendental Functions.* Cambridge University Press: Cambridge (1996).